1 \hspace{1em} \textbf{CSE 599I: Homework \#2 solutions}

1. (a) First, by Fréchet’s embedding, we can choose $X_n \subseteq \ell^d_\infty$ to be isometric to $(G_n, d_{G_n})$. Suppose that $f : G_n \to \ell^d_\infty$ is a $D$-embedding. By scaling, we may assume that $\|f\|_{\text{Lip}} \leq 1$ and $\|f^{-1}\|_{\text{Lip}} \leq D$.

Now, since $G_n$ is 3-regular, there exist a constant $\delta > 0$ such that for at least $n^2/10$ pairs of nodes $x, y \in G_n$, we have $d_{G_n}(x, y) \geq \delta \log n$, hence letting

$$S = \left\{ \{x, y\} : x, y \in G_n \text{ and } \|f(x) - f(y)\|_\infty \geq \frac{\delta \log n}{D} \right\},$$

we have $|S| \geq n^2/10$.

Let $\{f_i : G_n \to \mathbb{R}\}_{i=1}^d$ be the coordinates of the mapping $f$. Since $\|f\|_{\text{Lip}} \leq 1$, we have $\|f_i\|_{\text{Lip}} \leq 1$ as well. On the one hand, for every $\{x, y\} \in S$, there must exist some $i \in [d]$ for which $|f_i(x) - f_i(y)| \geq \frac{\delta \log n}{D}$. On the other hand, the number of pairs mapped $\frac{\delta \log n}{D}$ apart by $f_i$ is at most

$$n \cdot \left\{ x : |f_i(x) - \text{med}(f_i)| \geq \frac{\delta \log n}{2D} \right\} \leq n^2 h(G) \cdot \frac{\delta \log n}{2D} \leq n^2 \cdot n^{-O(1/D)}.$$

Thus we need at least $d \geq n^{O(1/D)}$ coordinates to handle all the pairs in $S$.

(b) Use the fact that, by Hölder’s inequality, $\ell^d_\infty$ is $O(1)$-equivalent to $\ell^d_1 + \log d$ for all $d \geq 1$.

(c) Write $w_i = \sum_{j=1}^{2^k} w_{ij} e_j$. Let $T : \mathbb{R}^{2^k} \to L_2$ be any linear operator which has $\|T\|_{\text{Lip}} \leq L$ and $\|T^{-1}\|_{\text{Lip}} \leq 1$. Assume first of all that $1 \leq p < 2$. Then,

$$2^{k(1+2/p)} = \sum_{i=1}^{2^k} \|w_i\|_p^2 \leq \sum_{i=1}^{2^k} \|T w_i\|_2^2 = \sum_{i=1}^{2^k} \left( \sum_{j=1}^{2^k} w_{ij} T(e_j) \right)^2 = \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} \langle w_i, w_j \rangle \langle T(e_i), T(e_j) \rangle = 2^k \sum_{j=1}^{2^k} \|T(e_j)\|_2^2 \leq 4^k \cdot L^2,$$

which implies that $L \geq 2^{k(1-p^{-1/2})} = \left( \frac{|A|-1}{2} \right)^{1/p-1/2}$. When $p > 2$ apply the same reasoning, with the inequalities reversed.

The final result follows by noting that the norms on $\ell^d_p$ and $\ell^d_q$ are equivalent up to $d^{1/2} - 1/2$.

2. (a) The upper bound here is obvious. For the lower bound, let $f : S_{2n+1} \to \ell^d_1$ be an isometric embedding of the $(2n + 1)$-star. Without loss, the center $c$ is mapped to the origin. Consider the embedding of two leaves $x$ and $y$: $f(x) = (x_1, \ldots, x_d)$ and $f(y) = (y_1, \ldots, y_d)$. Then,

$$2 = \|f(x) - f(y)\|_1 = \sum_{i=1}^d |x_i - y_i| \leq \sum_{i=1}^d |x_i| + |y_i| = \|f(x) - f(c)\|_1 + \|f(y) - f(c)\|_1 = 2$$

We conclude that the middle inequality is tight, hence for every two leaves $x, y$ and $i \in [d]$, we have $x_i y_i \leq 0$. This implies that for every coordinate $i \in [d]$, there can be at most two leaves whose $i$th coordinates are non-zero. On the other hand, obviously every leaf requires at least one non-zero coordinate. We conclude that $d \geq n$. 

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(b) In fact, we can get a monotone coloring with at most $1 + \log_2 \ell$ distinct colors on every root-leaf path, where $\ell$ is the number of leaves in $T$. Simply choose a root-leaf path $P$ such that upon the removal of $P$ from $T$, each remaining connected component contains at most $\ell/2$ of the original leaves. (This is easy to do—e.g. choose the $\lfloor \ell/2 \rfloor$th leaf in an in-order traversal of the tree.) Now color the remaining subtrees recursively.

(c) It is clear that $\Pr(\|F\|_{\text{Lip}} \leq s) = 1$ since it is easily checked that $F$ never expands any edge by more than $s$. Now fix $x, y \in T$ with least common ancestor $u \in T$, and let $s_1, \ldots, s_k$ and $t_1, \ldots, t_h$ be the set of maximal $\chi$-monochromatic segments on the path from $u$ to $x$ and $y$, respectively. Note that since $\chi$ is monotone, the color classes of these segments are all pairwise disjoint. First, we have

$$
\mathbb{E}[|F(x) - F(y)|] \leq s \cdot \left( \sum_{i=1}^{k} \text{len}(s_i) \mathbb{E}[\chi(s_i)] + \sum_{i=1}^{h} \text{len}(t_i) \mathbb{E}[\chi(t_i)] \right)
= \sum_{i=1}^{k} \text{len}(s_i) + \sum_{i=1}^{h} \text{len}(t_i) = d(x, y).
$$

On the other hand, observe that $\varepsilon_i$ has the same distribution as $|\varepsilon_i| \cdot \sigma_i$ where $\{\sigma_i\}_{i=1}^\infty$ is a family of i.i.d. $\pm 1$ Bernoulli random variables independent from the family $\{\varepsilon_i\}_{i=1}^\infty$. We use $\mathbb{E}_\sigma$ and $\mathbb{E}_\varepsilon$, respectively, to denote expectations over these random variables. Using Khintchine’s inequality, we have

$$
\mathbb{E}[|F(x) - F(y)|] = s \cdot \mathbb{E}_\varepsilon \left| \sum_{i=1}^{k} \text{len}(s_i) \varepsilon_{\chi(s_i)} + \sum_{i=1}^{h} \text{len}(t_i) \varepsilon_{\chi(t_i)} \right| \\
\approx s \cdot \mathbb{E}_\varepsilon \left| \sum_{i=1}^{k} \text{len}(s_i) \varepsilon_{\chi(s_i)}^2 + \sum_{i=1}^{h} \text{len}(t_i) \varepsilon_{\chi(t_i)}^2 \right|
$$

Now, we let $M = |\{i \in [k] : \varepsilon_{\chi(s_i)} \neq 0\}| + |\{i \in [h] : \varepsilon_{\chi(t_i)} \neq 0\}|$. Observe that $\mathbb{E} M = (k+h)/s$. Let $\mathcal{E}$ be the event that $M \leq \max \left\{ 1, \frac{2(k+h)}{s} \right\}$, and note that $\Pr(\mathcal{E}) \geq \frac{1}{2}$. Using Cauchy-Schwartz and $k, h \leq O(\log n)$, we have

$$
\mathbb{E}[|F(x) - F(y)|] \geq \Omega(1) \cdot s \cdot \Pr[\mathcal{E}] \cdot \mathbb{E} \left[ \sum_{i=1}^{k} \text{len}(s_i) \varepsilon_{\chi(s_i)}^2 + \sum_{i=1}^{h} \text{len}(t_i) \varepsilon_{\chi(t_i)}^2 \mid \mathcal{E} \right] \\
\geq \frac{\Omega(1) \cdot s}{\max\{1, \sqrt{\log n}/s\}} \left( \sum_{i=1}^{k} \text{len}(s_i) \mathbb{E}[\varepsilon_{\chi(s_i)}^2 \mid \mathcal{E}] + \sum_{i=1}^{h} \text{len}(t_i) \mathbb{E}[\varepsilon_{\chi(t_i)}^2 \mid \mathcal{E}] \right) \\
\geq \Omega(1) \cdot \min\left\{ 1, \sqrt{\frac{s}{\log n}} \right\} d(x, y),
$$

noting that $\mathbb{E}[\varepsilon_{\chi(s_i)}^2 \mid \mathcal{E}] = \mathbb{E}[\varepsilon_{\chi(t_i)}^2 \mid \mathcal{E}] \approx 1/s$.