Menu Complexity
for the Space Between
Single- and Multi-Dimensional
Mechanism Design

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Recap of Before the Break
3 items for sale

**Goal:** Determine who gets what and who pays what

**Identical:**
- “single-dimensional”

**All different:**
- “multi-dimensional”
  - Combinatorial valuations
  - Additive Valuations
  - Independent valuations

**Something in between?**
Taxation and Menus

**Menu**

- \((a(v_1), p(v_1))\)
- \((a(v_2), p(v_2))\)
- \((a(v_3), p(v_3))\)

Let buyer pick favorite

Buyer with \(v\) picks own option

**Question:** What size of menu is needed to guarantee revenue?

Incentive-compatibility (truthfulness): For all \(w\), \(u(v) > u(w \mid v)\)

Restricting to IC (truthful) mechanisms is without loss.
\( n \) items for sale

Identical:
- “single-dimensional”

All different:
- “multi-dimensional”
  - Combinatorial valuations
  - Additive Valuations
  - Independent valuations

Something in between?
Menu Complexity for Approximation: 1 buyer, additive over n independent items

1 \frac{1}{n} \text{ - approx}
LY '13

poly(n)

(1 - \varepsilon) - approx?

exp(n)

\left( 1 - \frac{1}{n} \right) \text{ - approx}
BGN '17

finite

\text{Constant - approx}

LB: HN '12
UB: BGN '17
BILW '14

infinite

Optimal

DDT '13
Multi-Dimensional Menu Complexity for \( n \) Items

1 item

Myerson ‘81

(1 − \( \varepsilon \))−approx for FedEx
SSW ‘18

FedEx
FGKK ‘16

Partially-Ordered
DGSSW ‘18

1

poly(n)

exp(n)

finite

infinite

\( \frac{1}{n} \)−approx

LY ‘13

Constant-approx

LB: HN ‘12
UB: BGN ‘17
BILW ’14

(1 − \( \varepsilon \))−approx?

(1 − \( \frac{1}{n} \))−approx

BGN ‘17

1 buyer,
n indep. items,
additive

Optimal
DDT ‘13
Optimal Menu Complexity Spectrum

1 2^{n-1} 3 \cdot 2^{n-1} - 1 \quad \text{unbounded} \quad \text{uncountable}

- **Single-dimensional**
  - 1 item
  - FedEx 1, 2, 3-day shipping

- **Partially-Ordered**
  - Wifi, +TV, +Cable

- **Budgets**
  - $5, $10, $12 budgets

- **“Multi-dimensional”**
  - 2 heterogenous items

- **unbounded**
  - Uncountable
Lower Bounds for $(1 - \varepsilon)$-approximations

1 \ - \ \Omega\left(\frac{1}{\sqrt{\varepsilon}}\right) \ - \ \Omega\left(\frac{1}{\varepsilon}\right) \text{ for } \varepsilon = \frac{1}{n^2} \ - \ \Omega\left(\exp\left(\frac{1}{\varepsilon}\right)\right) \text{ for } \varepsilon = \frac{1}{n} \ - \ \text{infinite}

- n independent additive items \ G '18
- n independent additive items \ BGN '17
- n correlated heterogenous items \ BCKW '10, HN '13

FedEx
SSW '18
The degree of complexity in the menu comes from the IC constraints which stitch together otherwise separate 1D problems.

Methods for understanding this:
- **Part I**: Revenue Curves
- **Part II**: Complementary Slackness conditions
Part I: Revenue Curves

METHODOLOGY FOR UNDERSTANDING THE NUMBER OF PRICES NEEDED
Mapping Prices to Revenue

\[ p \times \Pr[\text{customer buys at price } p] = p \cdot [1 - F_1(p)] \]
Allocation Rules and Prices

![Graph showing allocation rules and prices]

- **prob. of service**
- **value**

The graph illustrates the allocation of service based on certain values, indicating the probability of service under different conditions.
Allocation Rules and Prices

Menu

$$\left( 1, \frac{1}{3}p + \frac{2}{3} \bar{p} \right)$$

$$\left( \frac{1}{3}, \frac{1}{3}p \right)$$

$$\left( 0, 0 \right)$$

Menu Size = 2

= # prices in supp
Any allocation is a dist. over prices

\[ a(p) = \int_0^p \frac{d}{dv} a(v) dv \]
Randomized Pricings

\[ E[\text{rev}] \]

price

0

H
“Ironed” Revenue Curve

Least concave upper bound on curve (in value space)

price

E[rev]

OPT

0  H
The FedEx Setting

[FIAT GOLDNER KARLIN KOUSTOUPIAS 2016]
The FedEx Setting

value $v =$ how much shipping the package is worth

$\text{(v,i) } \sim F$

deadline $i$ = when they need the package sent by

*indifferent if the package is shipped early

[Fiat G. Karlin Koutsoupias ’16]
The FedEx Setting

\[(v, d) \sim F\]

Menu

\[
\begin{align*}
(1 \cdot 1, \$5) \\
\left(2 \cdot \frac{2}{3}, \$3\right) \\
(2 \cdot 1, \$5)
\end{align*}
\]

Mechanism

\[
a_i(v, d) = \text{prob buyer gets i-day shipping}
\]

\[
p(v, d) = \text{expected payment}
\]

Can have \textit{exponentially} many menu options

[Fiat G. Karlin Koutsoupias '16]
How do we maximize revenue for 2 days?

Set best prices for each day when they are decreasing.
How do we maximize revenue for 2 days?

Day 1 revenue curve

Day 2 revenue curve

Not incentive compatible!
FedEx
Revenue Curves
Constrained revenue from Day 2:

Price for day 1

E[rev]

price

0 H
Constrained revenue from Day 2:

\[
E[\text{rev}] \quad \text{Price for day 1} \quad \text{Price for day 1}
\]

0 \quad \text{price of day 1} \quad H

25
What to optimize:

E[rev] of Day 1 and Day 2
What to optimize:

- Day n-1 and n
- Day n

- E[rev]
- price of day n-2
What to optimize:

Day n-1 and n

Day n

E[rev]

price of day n-2

0 H
Optimal Variables
Optimal Allocation Rule

\[ a_2(v) \]

\[ E[\text{rev}] \]
Optimal Allocation Rule

\[ E[\text{rev}] \]

**Diagram:**
- Vertical axis: prob. of shipment
- Horizontal axis: bid
- Graph showing the allocation rule \( a_2(v) \) with different probabilities at different bids.
- Graph showing the expected revenue \( E[\text{rev}] \) over the price range from 0 to \( H \).
Bad Example
Exponential Menu Complexity

**Upper Bound:** In the worst case, each deadline $i$ has $2^{i-1}$ options. [Fiat G. Karlin Koutsoupias ’16]

**Lower Bound:** Distributions exist for this example, forcing $2^{i-1}$ options for each deadline. [Saxena Schwartzman Weinberg ’18]

Menu size is $2^n - 1$ overall, tight.
Approximate FedEx Menu Complexity

[SAXENA SCHVARTZMAN WEINBERG 2018]
Limiting Menu Complexity

How can we achieve good revenue with a small menu, or equivalently randomizing over fewer prices?

Idea: We only randomize over un-ironed peaks. What if we constrain this number?
Revenue via Polygon Approximation

Menu size is limited by the # points supporting the curve.

[Saxena Schwartzman Weinberg ’18]
Menu Complexity for (1 − ε)-approx

Upper Bound: $O \left( \frac{3}{n^2} \sqrt{\min\left\{ \frac{n}{\varepsilon}, \ln(H) \right\}} \right) = O \left( \frac{n^2}{\varepsilon} \right)$

Lower Bound: $\Omega(n^2) = \Omega \left( \frac{1}{\varepsilon} \right)$ for $\varepsilon = O \left( \frac{1}{n^2} \right)$
Revenue Curve Recap

- **Splitting** into multiple prices originates from IC constraints.

- Curves depict the limits of how prices can split.

- Essentially any combination of peaks/valleys can exist. [Saxena Schwartzman Weinberg ’18]

- When the mechanism is determined by revenue curves, approximation can be done via revenue curve approximation.
Part II: Duality Approach

METHODOLOGY FOR REASONING ABOUT WHEN ALLOCATION PROBABILITIES MUST BE DISTINCT
The Primal

Maximize \( E[\text{Rev}] \)

subject to:
- more utility for \((v, i)\) than \((v', i')\)
- feasibility

Maximize \( E[\text{Virtual Welfare}] \)

subject to:
- more utility for \((v, i)\) than \((v', i')\)
- weak monotonicity of allocation
- feasibility
Duality

**Primal**

maximize $f(x)$

subject to $g(x) \geq 0$

**Dual**

minimize $h(y)$

subject to $r(y) \leq 0$

Optimal pair $(x, y) \iff$ complementary slackness is satisfied, feasible: $g(x) = 0$ or $y = 0$; $h(y) = 0$ or $x = 0$.

**Lagrangian Primal:** maximize$_x$ minimize$_y$ $f(x) + y g(x)$.

**Lagrangian Dual:** minimize$_y$ maximize$_x$ $f(x) + y g(x)$.

Complementary slackness: $g(x) = 0$ or $y = 0$. 
The Primal

The Primal

\[ \text{maximize} \quad \sum_i \int_0^H f_i(v) \varphi_i(v) a_i(v) \, dv \]

subject to:

\[ \int_0^v a_i(x) \, dx - \int_0^v a_{i-1}(x) \, dx \geq 0 \]

\[ a'_i(v) \geq 0 \]

\[ a_i(v) \in [0, 1] \]

\[ \alpha_{i,i-1}(v) \quad \forall i \in \{2, \ldots, n\} \]

\[ \lambda_i(v) \quad \forall i, v \]

\[ \text{feasibility} \]

\[ \text{Report } i \text{ over } i' \]

\[ \text{Report } v \text{ over } v' \]

\[ a_i(v) := \text{Pr}[i\text{-day shipping to bidder with } (v, i)] \]

\[ = \text{E}[\text{rev}_i] \text{ using payment identity} \]
The Dual

minimize $\lambda, \alpha$ maximize feasible $a$

$$\sum_i \int_0^v f_i(v) a_i(v) \Phi_i(v) dv$$

where

$$\Phi_i(v) := \varphi_i(v) + \frac{(\int_v^H \alpha_{i,i-1}(x) \, dx - \int_v^H \alpha_{i+1,i}(x) \, dx)}{f_i(v)} - \lambda_i'(v)$$
An Optimal Primal/Dual Pair

\[ \text{minimize } \lambda, \alpha \quad \text{maximize } \text{feasible } a \]

\[ \sum_i \int_0^H f_i(\nu) a_i(\nu) \Phi_i(\nu) d\nu \]

Complementary Slackness:

**Constraint** is tight (= 0) or **dual variable** is 0.

- Report \( i \) over \( i' \)
  \[ \int_0^\nu a_i(x) dx - \int_0^\nu a_{i-1}(x) dx \geq 0 \]

- Report \( \nu \) over \( \nu' \)
  \[ a'_i(\nu) \geq 0 \]

Dual variables:

\[ \alpha_{i,i-1}(\nu), \lambda_i(\nu) \]

Can’t change \( \lambda, \alpha \) to further minimize.
Understanding Dual Variables
Virtual Values

It’s left to us to determine the allocation in the zeroes to satisfy complementary slackness.
Dual Variable $\alpha$ (reporting i over i-1)

$$\int_{0}^{v} a_i(x) \, dx - \int_{0}^{v} a_{i-1}(x) \, dx \geq 0$$

Report i over i-1 $\alpha_{i,i-1}(v)$

Day 1

$$f_1(v) \Phi_1(v)$$

Day 2

$$f_2(v) \Phi_2(v)$$

$$(v, 1)$$

$$\alpha_{2,1}(v) > 0$$

$$(v, 2)$$

Complementary Slackness:

Inter-day utility is equal ($u_1 = u_2$) where $\alpha_{2,1}$ is positive.
Dual Variable $\lambda$: (reporting $v$ over $v'$)

Complementary Slackness:
Utility is equal for reporting just under $v$—$a_i'(v) = 0$.
The allocation is constant in ironed intervals: $a_i(v) = a_i(y)$. 
Recap

Because $a$ maximizes $VW$,  
\[
\Phi_i(v) > 0 \implies a_i(v) = 1 \quad \text{and} \quad \Phi_i(v) < 0 \implies a_i(v) = 0
\]

Complementary slackness with $\lambda$:  
\[
\lambda_i(v) > 0 \text{ means } v \text{ is in an ironed interval } [v, v] \text{ and implies } a_i(v) \text{ is constant on } [v, v], \quad \text{or } a_i(v) = a_i(v).
\]

Complementary slackness with $\alpha$:  
\[
\alpha_{i,i-1}(v) > 0 \implies \text{utility is equal across deadlines } i, i-1
\]

\[
\begin{align*}
\Phi > 0 & \implies a = 1 \quad \text{and} \quad \Phi < 0 & \implies a = 0 \\
\lambda_i(v) & > 0 \implies \text{allocation constant} \\
\alpha_{i,i-1}(v) & > 0 \implies \text{utility of } i,i-1 \text{ equal at } v
\end{align*}
\]
Implications for the Primal

VIA COMPLEMENTARY SLACKNESS
Splitting the allocation

\[ \Phi > 0 \Rightarrow a = 1 \text{ and } \Phi < 0 \Rightarrow a = 0 \]

\[ \lambda_i(v) > 0 \Rightarrow \text{allocation constant} \]

\[ \alpha_{i,i-1}(v) > 0 \Rightarrow \text{utility of } i,i-1 \text{ equal at } v \]
FedEx Worst Case
FedEx Menu Complexity

- Exponentially many prices for day $i$ ($2^{i-1}$)
- Exponentially many prices total ($2^n-1$) [Fiat G. Karlin Koutsoupias ‘16]
- Proven to be tight. [Saxena Schwartzman Weinberg ’18]
The Budgets Setting

- **value** $v =$ how much the item is worth
- **budget** $B =$ how much they can afford

Result: At most $3 \cdot 2^{n-1} - 1$ prices.

[Devanur Weinberg ’17]
Partially-Ordered Items

[DEVANUR GOLDFNER SAXENA SCHVARTZMAN WEINBERG 2018]
The Partially-Ordered Setting

interest $G$
= service or set of goods desired

$(v,G) \sim F$

value $v$ = how much getting their interest is worth
An Optimal Primal/Dual Pair

minimize \( \lambda, \alpha \)  
maximize feasible \( a \)

\[
\sum_{G} \int_{0}^{H} f_{G}(v) a_{G}(v) \Phi_{G}(v) dv
\]

Complementary Slackness:

**Constraint** is tight (= 0) or **dual variable** is 0.

Report \( G \) over \( G' \)

\[
\int_{0}^{v} a_{G}(x) dx - \int_{0}^{v} a_{G'}(x) dx \geq 0
\]

Report \( v \) over \( v' \)

\[
a'_{G}(v) \geq 0
\]

Can't change \( \lambda, \alpha \) to further minimize.

For all \( G' \in N^{+}(G) \)

Dual variables

\[
\alpha_{G,G'}(v) \\
\lambda_{G}(v)
\]
Dual Variables and Virtual Values

Complementary Slackness:
Utility of G and G’ are equal where $\alpha_{G,G'} > 0$.

Implies e.g. if $\alpha_{c,A} > 0$ then A is at least as preferable as B.
Dual Variables and Virtual Values

Interest A

Interest B

\[ \alpha_{C,A}(v) > 0 \]

\[ \alpha_{C,B}(v) > 0 \]

\[ \Phi > 0 \implies a = 1 \text{ and } \Phi < 0 \implies a = 0 \]

\[ \lambda_G(v) > 0 \implies \text{allocation constant} \]

\[ \alpha_{C,A}(v) > 0 \implies A \text{ is preferable to } B \text{ at } v \]
Menu Complexity
Lower Bound
Key Idea for the Lower Bound

\[ a_A(x_1) > 0 \]
\[ a_B(x_1) = 1 \]

\[ a_A(r_A) > 0 \]

\[ \Phi > 0 \Rightarrow a = 1 \text{ and } \Phi < 0 \Rightarrow a = 0 \]

\[ \lambda_G(v) > 0 \Rightarrow \text{allocation constant} \]

\[ \alpha_{C,A}(v) > 0 \Rightarrow A \text{ is preferable to } B \text{ at } v \]
Key Idea for the Lower Bound

\[ a_A(x_1) > 0 \]
\[ a_A(r_A) > 0 \]

\[ \Phi > 0 \Rightarrow a = 1 \quad \text{and} \quad \Phi < 0 \Rightarrow a = 0 \]
\[ \lambda_G(v) > 0 \Rightarrow \text{allocation constant} \]
\[ \alpha_{C,A}(v) > 0 \Rightarrow A \text{ is preferable to } B \text{ at } v \]
Key Idea for the Lower Bound

\[ a_A(x_1) > 0 \]
\[ r_A \]
\[ a_A(r_{A}) > 0 \]
\[ r_A \]
\[ > a_B(r_{B}) \]

\[ a_B(x_1) = 1 \]
\[ a_B(x_2) > 0 \]
\[ a_B(r_{B}) > 0 \]

Φ > 0 ⇒ a = 1 and Φ < 0 ⇒ a = 0
\[ \lambda_G(v) > 0 ⇒ \text{allocation constant} \]
\[ \alpha_C(v) > 0 ⇒ A \text{ is preferable to } B \text{ at } v \]
Lower Bound

For any M:

\[ a_A(x_1) > 0 \]
\[ > a_B(x_2) \]
\[ a_A(x_3) > 0 \]
\[ > a_B(x_4) \]
\[ \vdots \]
\[ a_A(r_A) > 0 \]

\[ \Phi > 0 \Rightarrow a = 1 \text{ and } \Phi < 0 \Rightarrow a = 0 \]
\[ \lambda_G(v) > 0 \Rightarrow \text{allocation constant} \]
\[ \alpha_{C,A}(v) > 0 \Rightarrow A \text{ is preferable to } B \text{ at } v \]

> M different options are presented to the buyer.
Master Theorem (Informal)

For any dual that is given only by **signs** and **nonnegative variables** (ironed intervals + $\alpha$ flow), there exists a distribution that causes this dual.

**Corollary:**
The “bad dual” exists.
Menu Complexity
Upper Bound
Upper bound

A chain is a sequence of overlapping ironed intervals with $\alpha > 0$ at specific points.

If there are $M$ such intervals, the menu size is at most $2M – \text{finite.}$

If there are infinitely many intervals, they’re bounded and monotone, so they converge to a point that has virtual value 0 and is un-ironed for both A and B – menu size 1.

Always finite!
Multi-Unit Pricing
Lower Bound
The Multi-Unit Pricing Setting

value \( v \) = how much each item is worth

\((v,d) \sim F\)

demand \( d \) = how many units they want

# item options

1
2
3
\( n \)

[Devanur Haghpanah Psomas ’17]
Extension to MUP

At 2/3 payment

At 1/2 payment

\[ \bar{r}_A = \frac{x_1}{2} A \]

\[ \frac{x_1}{2} A = \frac{x_3}{2} A \]

\[ x_1 = \frac{x_3}{2} \]

\[ \frac{x_2}{2} A = \frac{x_4}{2} A \]

\[ x_2 = \frac{x_4}{2} \]

\[ \frac{x_{M-2}}{2} A = \frac{x_{M-1}}{2} A \]

\[ x_{M-2} = \frac{x_{M-1}}{2} \]

\[ \frac{x_{M-1}}{2} A = \bar{r}_A \]

\[ \bar{r}_B = \bar{x}_{2B} \]

\[ x_1 \]

\[ x_2 \]

\[ x_3 \]

\[ x_4 \]

\[ x_{M-2} \]

\[ x_{M-1} \]

\[ x_M \]
Summary
The Settings

Each buyer has a most-preferred-outcome (e.g. 3-day shipping).

The outcomes are structured such that a buyer’s value for this outcome tells you his value for all outcomes.

Properties:

- **Collapsible allocation rule**: degree of happiness
- **Reduced IC constraints**: specified by structure
- **Single-dimensional perks**: payment identity, etc
The Methods

Revenue Curves:
- Exactly where complexity grows or “splits”
- Limits of splitting
- Approximation via polygons

Complementary Slackness conditions:
- Where are certain outcomes preferred?
- Where must the allocation be positive?
- Where must the allocation be distinct, forcing different menu options?
- What are the limits to this?
Optimal Menu Complexity Spectrum

Single-dimensional
1 item

FedEx 1,2,3-day shipping

$5, $10, $12 budgets

Partially-Ordered Wifi, +TV, +Cable

“Multi-dimensional” 2 heterogenous items

1 2^{n-1} 3 \cdot 2^{n-1} - 1 unbounded uncountable
Lower Bounds for $$(1 - \varepsilon)$$-approximations

1. $\Omega\left(\frac{1}{\sqrt{\varepsilon}}\right)$
2. $\Omega\left(\frac{1}{\varepsilon}\right)$ for $\varepsilon = \frac{1}{n^2}$
3. $\Omega\left(\exp\left(\frac{1}{\varepsilon}\right)\right)$ for $\varepsilon = \frac{1}{n}$

Infinite

- n independent additive items
  - G '18
- n independent additive items
  - BGN '17
- n correlated heterogenous items
  - BCKW '10, HN '13

FedEx
SSW '18
Multi-Dimensional Menu Complexity for $n$ Items

- 1 item
  - Myerson '81

- $\frac{1}{n}$-approx
  - LY '13

- Constant-approx
  - LB: HN '12
  - UB: BGN '17
  - BILW '14

- $(1 - \varepsilon)$-approx for FedEx
  - SSW '18

- $(1 - \varepsilon)$-approx
  - (approx?)

- FedEx
  - FGKK '16

- Budgets
  - DW '17

- Partially-Ordered
  - DGSSW '18

- 1 buyer,
  - $n$ indep. items,
  - additive

- poly($n$)

- exp($n$)

- finite

- infinite

- (1 - $\frac{1}{n}$)-approx
  - BGN '17

- Optimal
  - DDT '13
Key Open Problems

- Other settings with more complex IC links?
- Lower bounds in terms of $\varepsilon$?
- Constant-factor approximations?
- Multiple bidders?
- Filling out the questions asked in Yannai’s talk in this setting.

\[ f_{c,A}(v) \quad f_{c,B}(v) \]
Thank you!