**STRUCTURED ESTIMATION OF GAUSSIAN GRAPHICAL MODELS**

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**PROBLEM**

- We consider the problem of jointly estimating multiple Gaussian graphical models.
- **Setting**: High-dimensional setting with more variables than samples.
- **Prior knowledge**: We assume prior knowledge on the structure of the GGMs—Specifically that the GGMs differ in node partitions. A node is said to be partitioned between two networks if the node has a high degree in the difference of networks.
- **Applications in gene-regulatory networks**: Regulatory genes play a prominent role in gene-regulatory networks. Detection of regulatory genes that differ between breast cancer and lung cancer gene-regulatory network is an important application.
- **Main contribution**: Propose a novel convex optimization based approach to detect node-based partitions in GGMs along with an efficient alternating direction method of multipliers (ADMM) algorithm.

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**PAST WORK**

- **Graphical Lasso [1]**: Single network estimation
  \[
  \min_{\Theta} \left\{ \|\Theta\|_1 \mid \sum_{i=1}^{p} \sum_{j=1}^{p} \Theta_{ij} = 0, \Theta_{ii} = 1 \right\}
  \]
  where $\Theta$ is sample-covariance matrix given by $X \sim \mathcal{N}(0, \Sigma)$ or $X$, $X_i, X_{i+1, j}, X_{j+1}$.
- **Fixed Graphical Lasso [GCL] [2]**: Multiple network estimation based on Edge based partitions
  \[
  \min_{\Theta} \left\{ \sum_{i=1}^{p} \sum_{j=1}^{p} \Theta_{ij} = 0, \Theta_{ii} = 1 \right\}
  \]
  where $\Theta(X_{i}, X_{j}) = \Theta(X_{i+1}, X_{j+1})$.

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**NAIVE APPROACH**

- **Support of difference of estimates expressed as complement of the union of sets of the union of sets**
- **Figure** depicts the true difference of networks and estimated difference of networks.

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**ROW-COLUMN OVERLAP NORM**

**Definition**: The row-column overlap norm (RCOM) induced by a matrix norm $\| \cdot \|$ is defined as:

\[
\|A\|_{\text{overlap}} = \sum_{i,j} |\langle A_{ij}, \Theta_{ij} \rangle |
\]

- **Variant of Overlap norm introduced by Jacob et al. [2]**
- **Accurate for symmetric graphs**
- **Generalize from norm used in [2] to arbitrary norms $\| \cdot \|$.

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**CONVEX FORMULATION - PNLG**

\[
\min_{\Theta} \left\{ \frac{1}{2} \|\Theta\|_1 - \frac{1}{2} \|\Theta\|_2^2 \right\}
\]

where $\Theta(X_{i}, X_{j}) = \Theta(X_{i+1}, X_{j+1})$.

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**POSSIBLE ESTIMATIONS**

- **Proximal operator computations for overlapping graph basis penalties [9] don’t apply since the RECON penalty promotes sum of overlapping graphs instead of the complete graph**.
- **Proposed proximal operator for RECON has not been done**. Also PNLG provides sparse solutions.

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**ADMM APPROACH**

- **Consider the following simple optimization problem**:

\[
\min_{g, X} \frac{1}{2} \|g(X) - X\|_1 + \frac{1}{2} \|X - X\|^2_{2}
\]

subject to $X = X$.

The ADMM algorithm [1] is as follows:

1. **Decide part of the objective to decompose**.
   - Here we decompose $g$ and $Y$ by introducing a new variable $Y$ and constraining $X = Y$.
   - The resulting optimization problem is given by:

\[
\min_{g, X, Y} \frac{1}{2} \|g(Y) - X\|_1 + \frac{1}{2} \|X - Y\|^2_{2}
\]

2. Form the augmented Lagrangian to (3) by first forming the Lagrangian and then augmenting it by a quadratic function of equality constraints. Lagrangian given by $L(Y, \lambda) = \frac{1}{2} \|g(Y) - X\|_1 + \frac{1}{2} \|X - Y\|^2_{2} + \sum_{i,j} \lambda_{ij} (X_{ij} - Y_{ij})$.

3. **Next minimize in turn each primal variable** keeping all other variables fixed. The dual variables get updated using a dual-ascent update.

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**REFERENCES**

