The W_1 state is the unique ground-state of a local Hamiltonian

Chinmay Nirkhe
University of Washington, *Seattle, Wash.*

A recent work by Gioia and Thorngren [1], shows that the W_1 state,

$$|W_1\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \underbrace{|0\rangle \dots |0\rangle}_{j-1 \text{ times}} |1\rangle \underbrace{|0\rangle \dots |0\rangle}_{n-j \text{ times}} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j |0^n\rangle , \tag{1}$$

cannot be the *unique* ground-state of any 1-dimensional local Hamiltonian. In this short note, we show that there exists a spectrally-gapped 2-local Hamiltonian $\mathbf{H} = \mathbf{E}_{ij}\, h_{ij}$ with all-to-all connectivity for which the W_1 state *is* the unique ground-state. Moreover, the local Hamiltonian term h_{ij} for $i \neq j \in [n]$ is the same term and is defined for a choice of $\theta \in (0, 1)$:

$$h_{ij} = h(i, j) \stackrel{\text{def}}{=} \frac{\theta}{n-2} \cdot |00\rangle\langle 00| + 1 \cdot |\Psi^-\rangle\langle\Psi^-| + 1 \cdot |11\rangle\langle 11| \tag{2}$$

where $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are two of the four Bell states.

Theorem 1. *The 2-local Hamiltonian*[1] *given by all-to-all connectivity and interaction h given by eq. (2) has a* unique *ground-state of* $|W_1\rangle$ *at energy* $\frac{\theta}{n}$ *and a spectral gap of at least* $\frac{2\theta}{n(n-1)}$.

Proof. To prove this statement, we can define a generalization of the W_1 state to ℓ excitations:

$$|W_\ell\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{\binom{n}{\ell}}} \sum_{L \subset [n] : |L| = \ell} X_L |0^n\rangle \tag{3}$$

where $X_L = \prod_{j \in L} X_j$. A simple calculation reveals that

$$|W_\ell\rangle = \sqrt{\frac{(n-\ell)(n-\ell-1)}{n(n-1)}} |00\rangle |W_\ell\rangle + \sqrt{\frac{2\ell(n-\ell)}{n(n-1)}} |\Psi^+\rangle |W_{\ell-1}\rangle + \sqrt{\frac{\ell(\ell-1)}{n(n-1)}} |11\rangle |W_{\ell-2}\rangle \tag{4}$$

and therefore, we can calculate the 2-qubit reduced density matrix of $|W_\ell\rangle$ directly:

$$\mathrm{tr}_{-[2]}(|W_\ell\rangle\langle W_\ell|) = \frac{(n-\ell)(n-\ell-1)}{n(n-1)} |00\rangle\langle 00| + \frac{2\ell(n-\ell)}{n(n-1)} |\Psi^+\rangle\langle\Psi^+| + \frac{\ell(\ell-1)}{n(n-1)} |11\rangle\langle 11| . \tag{5}$$

Then, the energy of $|W_1\rangle$ with respect to $\mathbf{H}$ is

$$\langle W_1|\mathbf{H}|W_1\rangle = \mathop{\mathbf{E}}_{i,j} \mathrm{tr}\,(h_{ij} |W_1\rangle\langle W_1|) = \mathop{\mathbf{E}}_{i,j} \mathrm{tr}\left(h\, \mathrm{tr}_{-[2]}(|W_1\rangle\langle W_1|)\right) = \frac{\theta}{n-2} \cdot \frac{n-2}{n} = \frac{\theta}{n}. \tag{6}$$

It remains to show that any orthogonal vector to $|W_1\rangle$ has energy larger than θ/n with respect to $\mathbf{H}$. First notice that $\mathbf{H} = P(\pi)^{-1}\mathbf{H}P(\pi)$ for any $\pi \in S(n)$. Therefore,

$$\langle\psi|\mathbf{H}|\psi\rangle = \mathop{\mathbf{E}}_{\pi} \mathrm{tr}\big(P(\pi)^{-1}\mathbf{H}P(\pi) |\psi\rangle\langle\psi|\big) = \mathop{\mathbf{E}}_{\pi} \mathrm{tr}\big(\mathbf{H}P(\pi) |\psi\rangle\langle\psi| P(\pi)^{-1}\big). \tag{7}$$

[1] The frustration in this Hamiltonian is necessary given that the reduced density matrix of the W_1 state.

Therefore, the energy of a state ψ equals that of its symmetrized version $\mathbf{E}_\pi P(\pi)^{-1}\psi P(\pi)$. Since the symmetrized state is supported on the symmetric subspace, there exists a symmetric state of equal energy. Therefore, it suffices to show that every orthogonal vector to $|W_1\rangle$ within the symmetric subspace has energy larger than θ/n.

Recall that the symmetric subspace for qubits is spanned by the states $|W_0\rangle, |W_1\rangle, |W_2\rangle, \ldots, |W_n\rangle$. Given the orthogonality of the W states of different excitations and the expansion in the basis of h given in eq. (4), it is easy to verify that $\langle W_{\ell'}| h |W_\ell\rangle = 0$ if $\ell \neq \ell'$. Equivalently, the off-diagonal terms in energy calculations in the symmetric subspace are 0. This implies that the energy of the Hamiltonian h must be minimized for some $|W_\ell\rangle$ as the energy of any state $\sum \alpha_\ell |W_\ell\rangle$ will be equal to that of the density matrix $\sum |\alpha_\ell|^2 |W_\ell\rangle\langle W_\ell|$.

We generalize our energy calculation of $|W_1\rangle$ in eq. (6), to calculate that

$$\langle W_\ell|\mathbf{H}|W_\ell\rangle = \frac{\theta}{n-2} \cdot \frac{(n-\ell)(n-\ell-1)}{n(n-1)} + \frac{\ell(\ell-1)}{n(n-1)}. \tag{8}$$

It is easy to see that this decreases as ℓ decreases. Therefore, $|W_2\rangle$ is the eigenvector of minimal eigenvalue orthogonal to $|W_1\rangle$. Given the eigenvalue of $|W_2\rangle$ due to the previous equation, the spectral gap of $\mathbf{H}$ is at least

$$\frac{\theta}{n} \cdot \frac{(n-3)+4/\theta}{n-1} - \frac{\theta}{n} \geq \frac{2\theta}{n(n-1)}. \tag{9}$$

□

No "better" 2-local Hamiltonian exists

The issue with the local Hamiltonian term h defined in eq. (2) is that the ratio of eigenvalues of h scale linearly in n. Is it possible to construct a 2-local term without such scaling? This is in fact impossible. Let us restrict ourselves to Hamiltonians h with spectrum bounded $\in [0,1]$.

To see this, consider a 2-local Hamiltonian $\mathbf{H}'$ with the property that

$$\langle W_1|\mathbf{H}'|W_1\rangle < \min\left\{ \langle W_0|\mathbf{H}'|W_0\rangle, \langle W_2|\mathbf{H}'|W_2\rangle, \langle W_3|\mathbf{H}'|W_3\rangle, \ldots \right\}. \tag{10}$$

Notice that for any such $\mathbf{H}'$, the Hamiltonian $\mathbf{H}'' \stackrel{\text{def}}{=} \mathbf{E}_{\pi\in S(n)} P(\pi)^{-1}\widetilde{\mathbf{H}}P(\pi)$ has the property that $\langle W_\ell|\mathbf{H}''|W_\ell\rangle = \langle W_\ell|\mathbf{H}'|W_\ell\rangle$ and $\mathbf{H}''$ is 2-local Hamiltonian where every local term is the same. Therefore, without loss of generality, we can restrict to considering such Hamiltonians $\mathbf{H}' = \mathbf{E}_{ij} h'(ij)$ Given the local interaction term h', define

$$E_0 \stackrel{\text{def}}{=} \langle 00|h'|00\rangle, \quad E_1 \stackrel{\text{def}}{=} \langle \Psi^+|h'|\Psi^+\rangle, \quad E_2 \stackrel{\text{def}}{=} \langle 11|h'|11\rangle. \tag{11}$$

Then, the equations $\langle W_1|\mathbf{H}'|W_1\rangle < \langle W_0|\mathbf{H}'|W_0\rangle$ and $\langle W_1|\mathbf{H}'|W_1\rangle < \langle W_2|\mathbf{H}'|W_2\rangle$ are equivalent to

$$\frac{n-2}{n}E_0 + \frac{2}{n}E_1 < E_0 \tag{12a}$$

$$\frac{n-2}{n}E_0 + \frac{2}{n}E_1 < \frac{(n-2)(n-3)}{n(n-1)}E_0 + \frac{4(n-2)}{n(n-1)}E_1 + \frac{2}{n(n-1)}E_2. \tag{12b}$$

These are equivalent to

$$E_1 < E_0 \tag{13a}$$

$$E_0 < \left(\frac{n-3}{n-2}\right) E_1 + \frac{1}{n-2} E_2. \tag{13b}$$

To minimize the energy of W_1 and maximize the energy of W_0 and W_2 with respect to h, it is optimal to set $E_2 = 1$. Setting E_1 to be anything > 0 is equivalent to adding a global energy penalty, so we can without loss of generality assume $E_1 = 0$ which forces $E_0 < \frac{1}{n-2}$. This is precisely what we show in our definition of h in eq. (2).

1 W states of two different types of excitations

Consider a qudit system of local dimension $d+1$ with a basis $|0\rangle, |1\rangle, \ldots, |d\rangle$ for the local dimension. A natural generalization of a W state to a qudit system could be

$$\left|W_1(\vec{a})\right\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \underbrace{|0\rangle \ldots |0\rangle}_{j-1 \text{ times}} |a_j\rangle \underbrace{|0\rangle \ldots |0\rangle}_{n-j \text{ times}} \tag{14}$$

where $\vec{a} \in \{1, 2, \ldots, d\}^n$ is a description of the excitation type. For any particular choice of $\vec{a}$, this state is the ground-state of a local Hamiltonian by simply adjusting the previous argument. Furthermore, we could even construct a local Hamiltonian whose ground-state enforces a particular superposition over excitations at site j.

But it seems hard to generate a local Hamiltonian which allows in its ground-space any state $\left|W_1(\vec{a})\right\rangle$.

References

[1] Lei Gioia and Ryan Thorngren. W state is not the unique ground state of any local hamiltonian, 2023.