Circuit lower bounds for low-energy states of code Hamiltonians

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The complexity of low-energy states

Groundstates of local Hamiltonians can be proven to be highly-entangled, non-classical, and capture universal q. computation.

But when we move past the groundspace to higher energy states, what is known? How entangled and non-classical are states of low-energy?

The complexity of low-energy states

- The decade old NLTS conjecture [Freedman-Hastings '14] states that $\exists$ local Hamiltonians for which every state of low-energy is highly-entangled and non-classical.
  - These local Hamiltonians have no classical approx. for ground-states
  - A state brought to "room-temp" of this Hamiltonian is still highly-entangled
- This work proves a circuit-depth lower-bound for every low-energy state of local Hamiltonians arising from error-correcting codes

Our Results

Let $\mathcal{C}$ be a $[[n, k, d]]$ stabilizer error-correcting code of const. locality.

(double sided LDPC)

n: # of physical qubits

k: # of logical qubits

d: erasure distance

Let H be the corresponding local Ham. $H = \sum H_i$ with $H_i = \frac{\mathbb{I} - C_i}{2}$.

C_i: check term of $\mathcal{C}$.

Let ρ be a mixed state s.t. $\operatorname{tr}(H\rho) \le \varepsilon n$. Then, the circuit depth of ρ

is at least $\Omega\left(\min\left\{\log d, \log\left(\frac{k}{n} \cdot \frac{1}{\varepsilon^{1.01}}\right)\right\}\right)$.

Our Results

Let $\mathcal{C}$ be a $[[n, k, d]]$ stabilizer error-correcting code of const. locality.

Let ρ be a mixed state s.t. $\mathrm{tr}(H\ \rho) \le \varepsilon n$. Then,

$$cc(\rho) \ge \Omega\left(\min\left\{\log d,\ \log\left(\frac{k}{n} \cdot \frac{1}{\varepsilon^{1.01}}\right)\right\}\right).$$

Cor If $k = \Omega(n)$ (linear rate) and $d = \Omega(n^c)$ (polynomial distance)

then for $\mathrm{tr}(H\ \rho) \le O(n^{0.99})$,

$$cc(\rho) \ge \Omega(\log n)$$

$\mathrm{tr}(H\ \rho) \le o(n)$

$$cc(\rho) \ge \omega(1).$$

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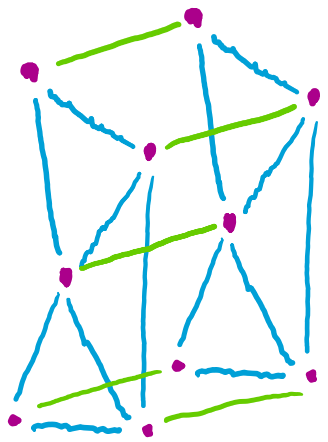
$cc(\rho) \ge \Omega(\log n)$ | $cc(\rho) \ge \omega(1)$.

Prior work: • $\omega(1)$ bound for all states of energy $\le O(n^{-2})$. [Kitaev '93]

• $\Omega(\log n)$ bounds for restricted subclasses of low-energy states: ε-error [Eldar-Harrow '17, N.-Vazirani-Yuen '18], one-sided error [Freedman-Hastings '14], Gibbs states [Eldar '19], $\mathbb{Z}_2$ symm. [Bravyi et al. '19]

Example codes

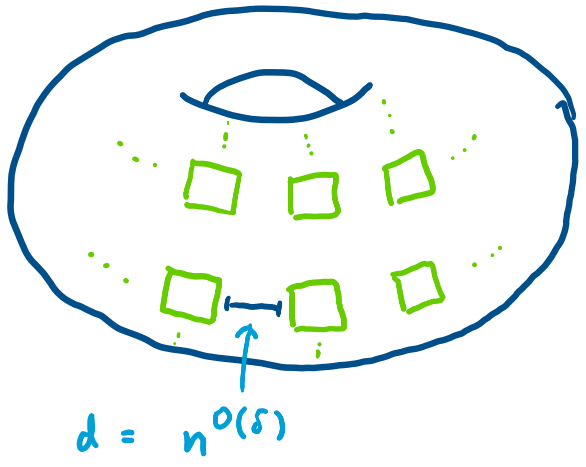
① Tillich-Zémor'09 hypergraph product codes



$k = \Theta(n)$

$d = \Theta(\sqrt{n})$

② Punctured toric code with $\Omega(n^{1-\delta})$ holes.



$k = \Theta(n^{1-\delta})$

$d = \Theta(n^{\delta})$

for $\mathrm{tr}(H\rho) \leq O(n^{1-2\delta})$, $cc(\rho) \geq \Omega(\delta \log n)$.

Progress towards the NLTS Conjecture

Conjecture [Freedman-Hastings[14]]: $\exists$ an $\varepsilon > 0$ and a family of local Hamiltonians $\{H^{(n)}\}$ on n qubits s.t. every state of energy $< \varepsilon n$ has a super-constant circuit-depth lower bound.

Our theorem proves a bound of $\tilde{\Omega}(\log(\frac{1}{\varepsilon}))$ vs superconstant.

- NLTS is a necessary consequence of the QPCP conjecture
- NLTS also makes progress on whether we can conduct quantum computations at room temperature.

Lower-bounds for low-energy states

What do low-energy states ($\leq \varepsilon n$) look like?

For one, they contain all states $< \varepsilon$ distance from the code.

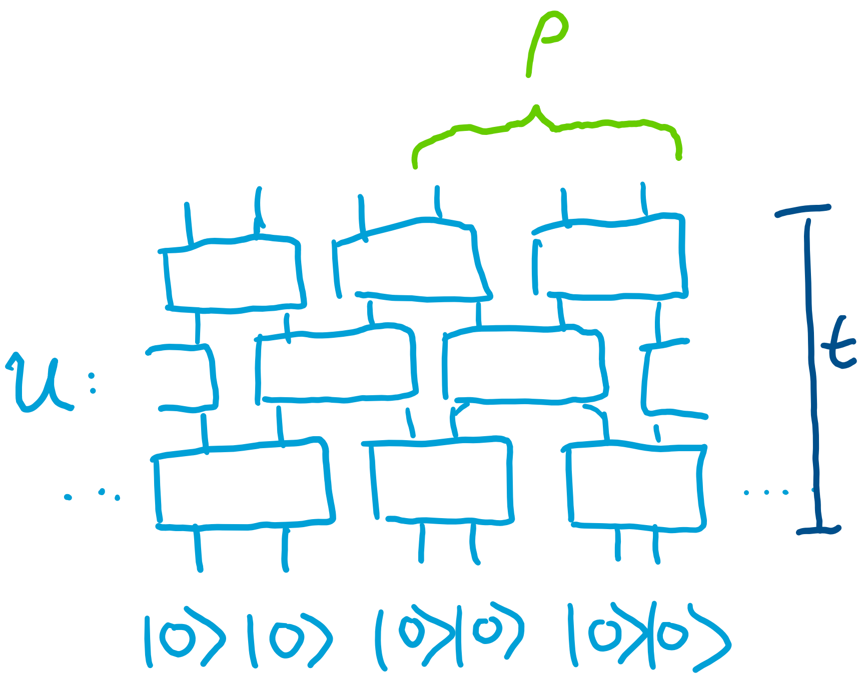
$$B_\varepsilon(\mathcal{C}) = \{ \psi : \exists \rho \in \mathcal{C} \text{ s.t. } \|\psi - \rho\|_1 < \varepsilon \}.$$

Pf. For $\psi \in B_\varepsilon(\mathcal{C})$,

$$\mathrm{tr}(H\psi) = \sum_i \mathrm{tr}(H_i \psi) \leq \sum_i \mathrm{tr}(H_i \rho) + \varepsilon = \sum_i \varepsilon \leq \varepsilon n.$$

Necessary Defs & Lemmas

Circuit Complexity

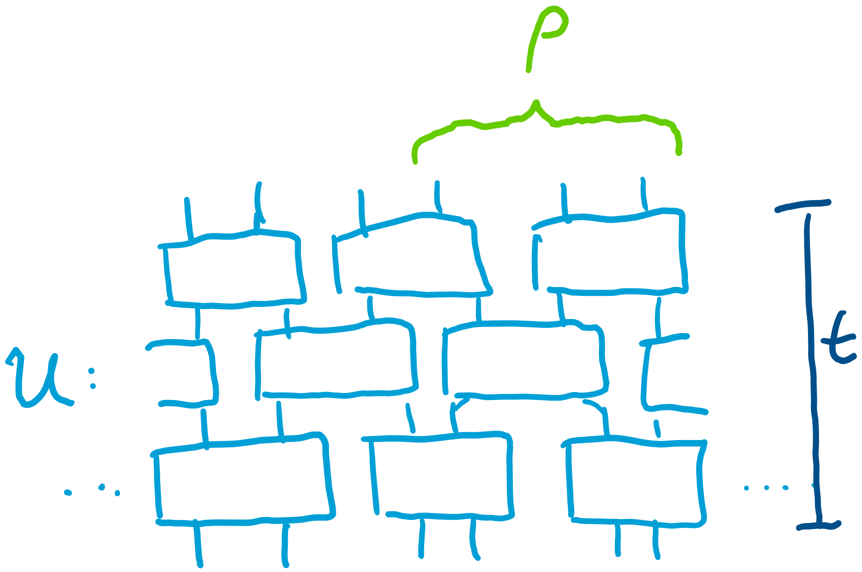


$cc(\rho)$ = min depth t of any ckt exactly producing ρ.

Fact A state has $cc \leq 1$ iff it is a tensor product state.

Fact Given a $O(1)$-local Ham. H and a state ρ of $cc(\rho) = t$, there is a classical alg. for computing $tr(H\rho)$ (i.e. energy) in time $poly(n) \cdot \exp(\exp(t))$.

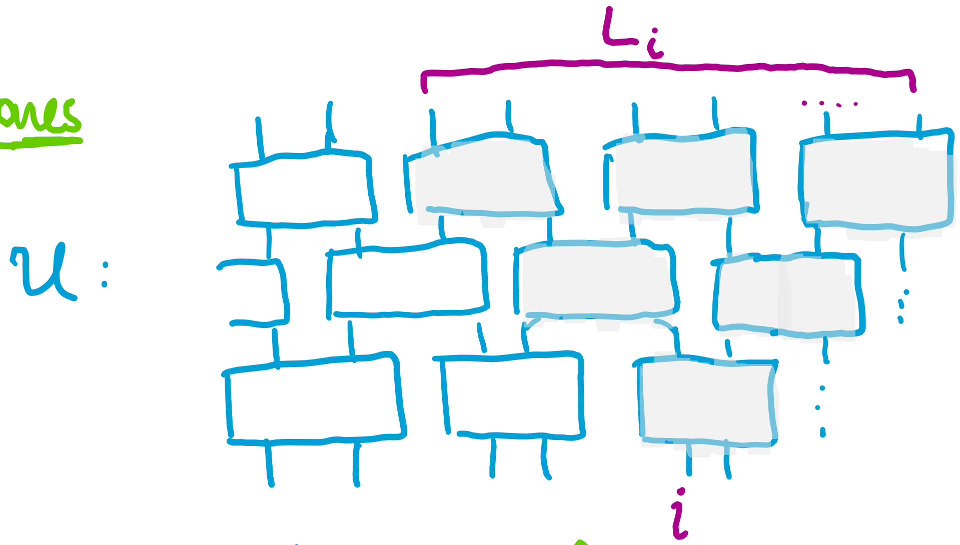
Circuit Complexity



$|0\rangle|0\rangle \; |0\rangle|0\rangle \; |0\rangle|0\rangle$

$cc(\rho)$ = min depth t of any ckt exactly producing ρ.

Lightcones



Fact 1 $|L_i| < 2^t$ ← depth of ckt.

reduced density matrix on L_i.

Fact 2 $tr_{-i}(U \rho U^\dagger) = tr_{-i}(U \rho_{L_i} U^\dagger)$

all qubits but i.

Error-correcting Codes

i.e. Local Indistinguishability.

An error-correcting code has distance d if for every codestate ρ and S a set of qubits of $|S| < d$,

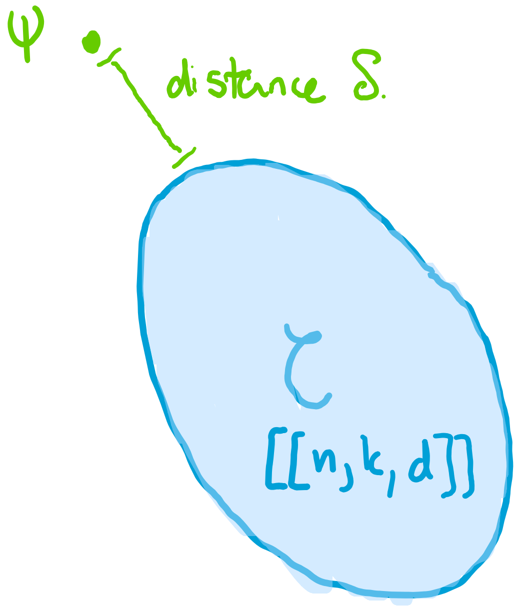
the state ρ can be recovered from $\rho_{-S} = \mathrm{tr}_S(\rho)$.

Equivalently, the state $\rho_S = \mathrm{tr}_{-S}(\rho)$ is an invariant over codestates ρ.

Pf. If ρ_S even partially depended on ρ, then this violates no-cloning.

Proof Sketch

Circuit LBs for low-distance states



Let ψ be a state dist δ from $\mathcal{C}$. What is $cc(\psi)$?

Folklore: For any codestate ρ, $cc(\rho) = \Theta(\log d)$.

For simplicity, let's only consider pure states and circuits without ancillas.

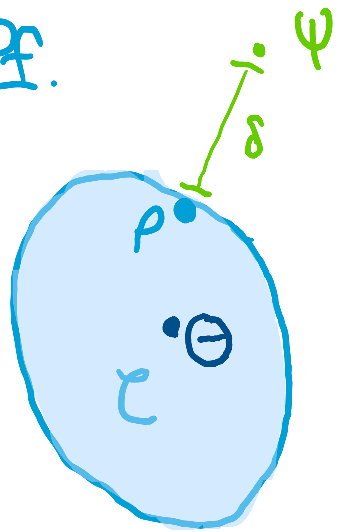
<u>Thm</u>: Let $\sqrt{\delta} < k/n$. for any state $|\psi\rangle$ of dist δ from $\mathcal{C}$.

$$\Rightarrow cc(\psi) \geq \Omega(\log d).$$

Circuit LBs for low-distance states

Thm : Let $\sqrt{\delta} < k/n$, for any state $|\psi\rangle$ of dist δ from $\mathcal{C}$. $\Rightarrow cc(\psi) \geq \Omega(\log d)$.

Pf. ρ be closest codestate to ψ. Θ be encoded maximally mixed state.



① Let R be a region of $|R| < d$.

$$\underbrace{\psi_R \approx_\delta \rho_R}_{\text{distance}} \underbrace{=}_{\text{local indistinguishability}} \Theta_R.$$

② Let $|\psi\rangle = \mathcal{U}|0\rangle^{\otimes n}$ for $\mathcal{U}$ of depth t s.t. $2^t < d$.

$|0\rangle\langle 0| = \mathrm{tr}_{-i}(\mathcal{U}^\dagger \psi \mathcal{U})$.

③ $|0\rangle\langle 0| \underset{②}{=} \mathrm{tr}_{-i}(\mathcal{U}^\dagger \psi \mathcal{U}) = \mathrm{tr}_{-i}(\mathcal{U}^\dagger \psi_{L_i} \mathcal{U}) \underset{①}{\approx_\delta} \mathrm{tr}_{-i}(\mathcal{U}^\dagger \Theta_{L_i} \mathcal{U}) = \mathrm{tr}_{-i}(\mathcal{U}^\dagger \Theta \mathcal{U})$.

[magenta ='s are from the lightcone argument]

④ $S(\mathrm{tr}_{-i}(\mathcal{U}^\dagger \Theta \mathcal{U})) \leq \sqrt{\delta}$. ⑤ $k = S(\Theta) = S(\mathcal{U}^\dagger \Theta \mathcal{U}) \leq \sum_{i=1}^{n} S(\mathrm{tr}_{-i}(\mathcal{U}^\dagger \Theta \mathcal{U})) \leq \sqrt{\delta}\, n$. $\perp$.

The set of low-energy states

Combinatorial states:

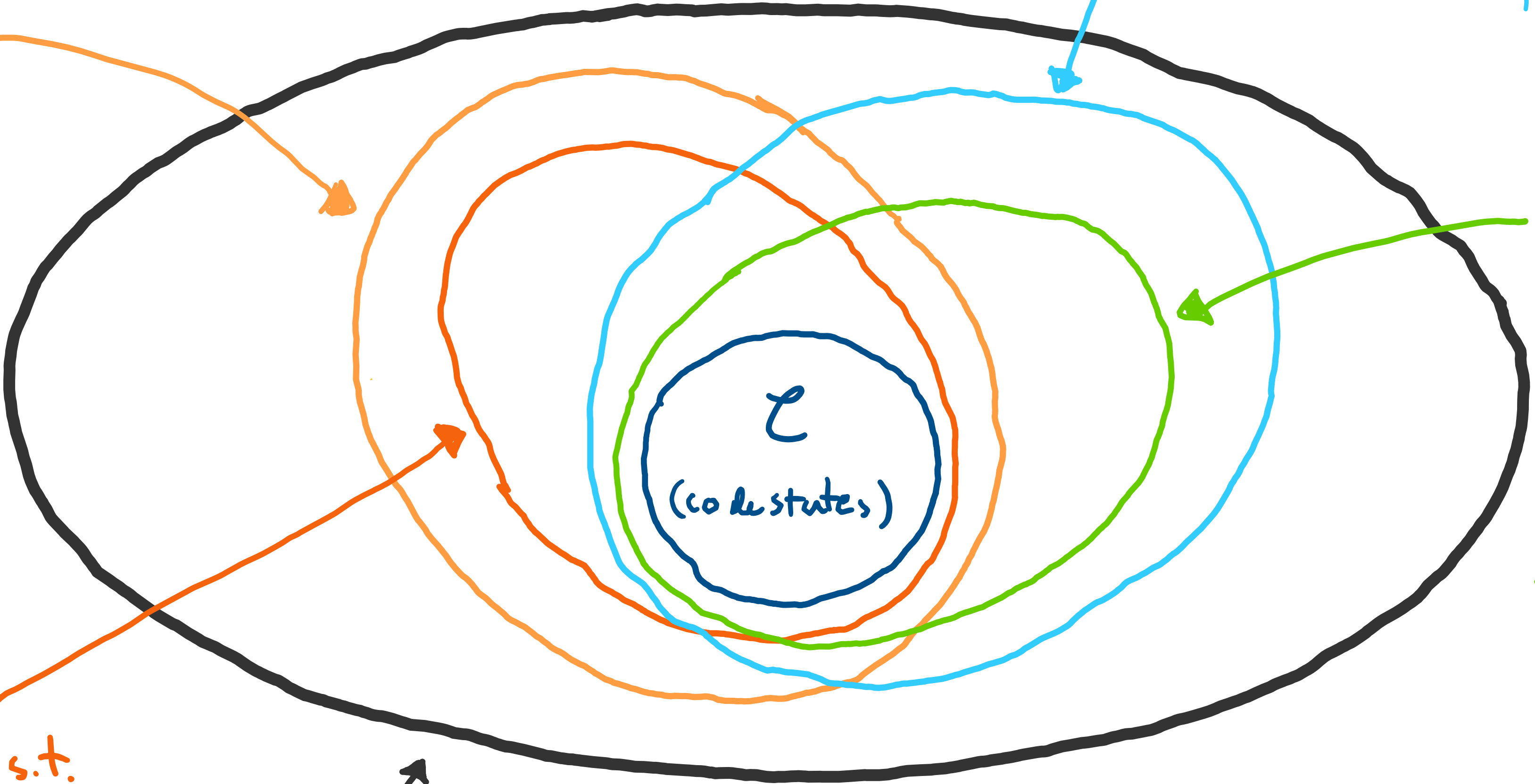
$\Pr_i \left(\mathrm{tr}(H_i \rho) \neq 0 \right) \leq \varepsilon.$

ε-smooth states: $\forall i,$

$\mathrm{tr}(H_i \rho) \leq \varepsilon.$

low-error states:

take codestate ψ and change up to $O(\varepsilon n)$ qubits



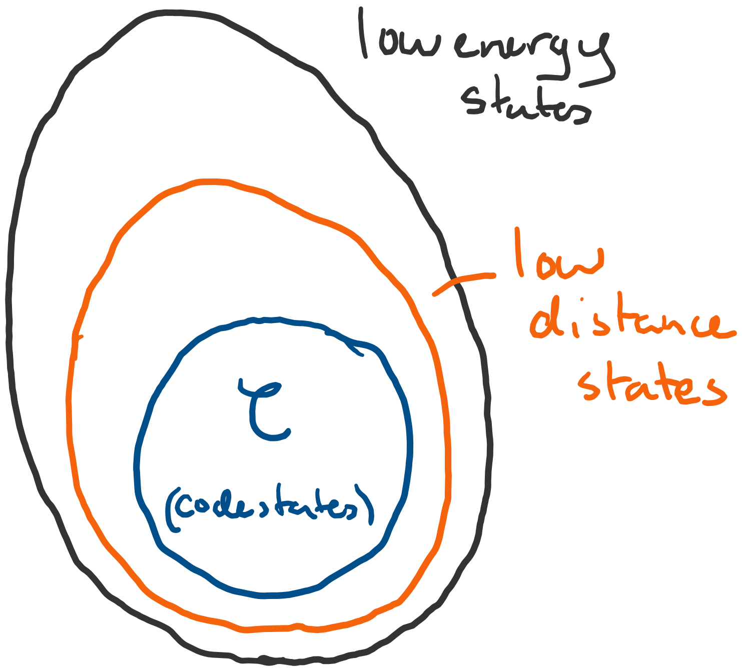
ε-dist states:

$\exists$ codestate ψ s.t.

$\| \rho - \psi \|_1 \leq \varepsilon.$

all low energy states: $\mathrm{tr}(H\rho) \leq \varepsilon n$

Extending the argument to low-energy states



If $k = \Omega(n)$, then circuit LBs for all low-distance states.

LDPC Stabilizer Codes

All checks are tensor products of a few Paulis.

$$\mathcal{C} = \{ |\psi\rangle : C_i |\psi\rangle = |\psi\rangle \ \forall i \}.$$

$$D_s = \{ |\psi\rangle : C_i |\psi\rangle = (-1)^{s_i} |\psi\rangle \ \forall i \} \text{ for } s \in \{0,1\}^m.$$

Rmk: Local indistinguishability holds for each eigenspace D_s: Region R s.t. $|R| < d$. Then, ρ_R invariant over each D_s.

(but can depend on s).

Extending the argument to low-energy states

Pf sketch:

① For a state ψ of energy $\mathrm{tr}(H\psi) \le \varepsilon n$, measure ψ with stabilizer code checks to collapse it into a mixture of eigenstates

② Apply an argument similar to the low-dist. argument in every eigenstate to conclude a bound on the rate of the code based on the circuit depth of ψ.

Open questions

- Is the combinatorial NLTS conjecture true for $\varepsilon = \Omega(1)$? (Eldar-Harrow'17)

  (each Hamiltonian term is either violated or not).

- Is the local Hamiltonian problem with promise gap $n^{-0.01}$ also QMA-hard?

  Our almost NLTS theorem is the corresponding statement.

- Circuit lower-bounds for other notions of groundspace approximation have been proven (such as NLETS [Eldar-Harrow'17, N.-Vazirani-Yuen'18])

  Can our techniques reproduce these results?

Thank you & any questions?

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