Circuit lower-bounds for low-energy states of quantum code Hamiltonians

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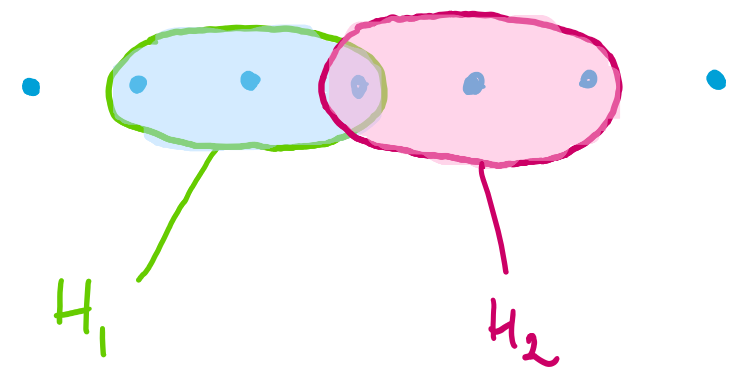
The QPCP Conjecture

n qubits:



Local Hamiltonian $H = \sum_{i=1}^{O(n)} H_i$

acting on n qubits.

$$E = \inf_{\phi} \operatorname{tr}(H\phi).$$

Given $\{H_i\}$, how hard is it to approximate E up to accuracy $\varepsilon(n)$?

Thm For $\varepsilon(n) = \frac{1}{n^2}$, its QMA-hard.

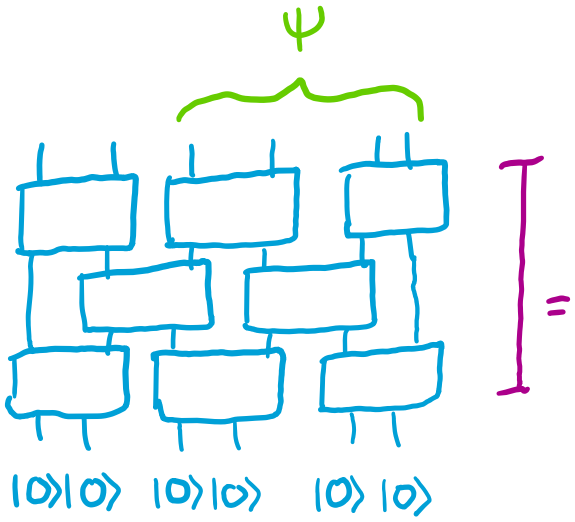
[Kitaev[03], Cahaylandau, Nagaj[18], Bausch, Crosson[18]]

QPCP Conjecture It is also QMA-hard for $\varepsilon(n) = \Omega(n)$.

[Aharonov, Naveh[02], Aharonov, Arad, Vidick[13]]

Simplifying the problem: NLTS Conjecture [Freedman, Hastings '14]

(No low-energy trivial states conj.)



$cc(\psi)$

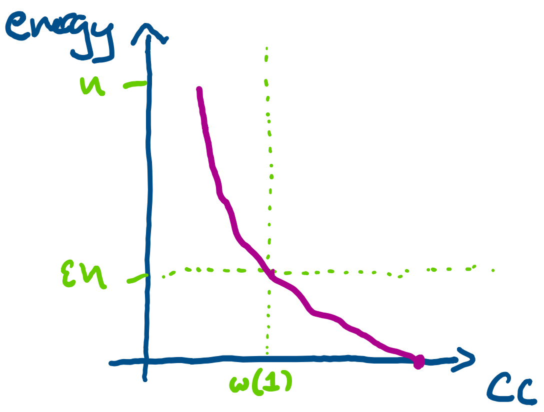
$=$ min depth*

* : gates are any 2 qubit unitaries do not need to be on geometrically local circuits

NLTS Conj. $\exists$ a fixed constant $\varepsilon > 0$, and a fam of local Ham. $\{H^{(n)}\}_n$ on n qubits s.t.

$\forall\ \psi^{(n)}$ with $\mathrm{tr}(H^{(n)}\psi^{(n)}) < \varepsilon n$,

the $cc(\psi^{(n)}) = \omega(1)$ (superconstant).



The NLTS Conjecture [Freedman, Hastings '14]

1. Necessary consequence of the QPCP conjecture

2. Seperates the "robustness of entanglement" question from the "hardness of computation" aspect of QPCP

3. Asks about the ability to conduct quantum computation at room temperature

Our Results

(double sided LDPC)

Let $\mathcal{C}$ be a $[[n, k, d]]$ stabilizer error-correcting code of const. locality.

# of physical qubits

# of logical qubits

erasure distance

Let $H_{\mathcal{C}}$ be the corresponding local Ham. $H_{\mathcal{C}} = \sum H_i$ with $H_i = \frac{\mathbb{I} - C_i}{2}$.

Let ρ be a mixed state s.t. $\mathrm{tr}(H_{\mathcal{C}}\,\rho) \le \varepsilon n$. Then,

$$cc(\rho) \geq \Omega\left(\min\left\{\log d, \log\left(\frac{k}{n} \cdot \frac{1}{\varepsilon \log(1/\varepsilon)}\right)\right\}\right).$$

An almost linear NLTS Theorem. (dependence on ε).

Our Results

QCMA $\neq$ QMA + QPLP $\Rightarrow$
all $\leq \varepsilon n$ energy states have superpoly cc.

Let $\mathcal{C}$ be a $[[n,k,d]]$ stabilizer error-correcting code of const. locality.

Let ρ be a mixed state s.t. $\mathrm{tr}(H_{\mathcal{C}}\,\rho) \leq \varepsilon n$. Then,

$$cc(\rho) \geq \Omega\left(\min\left\{\log d,\ \log\left(\frac{k}{n}\cdot\frac{1}{\varepsilon\log(1/\varepsilon)}\right)\right\}\right).$$

Cor If $k = \Omega(n)$ (linear rate) and $d = \Omega(n^c)$ (polynomial distance)

then for $\mathrm{tr}(H_{\mathcal{C}}\,\rho) \leq O(n^{0.99})$,

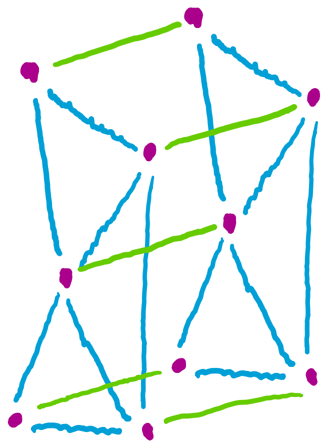
$$cc(\rho) \geq \Omega(\log n)$$

| $\mathrm{tr}(H_{\mathcal{C}}\,\rho) \leq o(n)$

$$cc(\rho) \geq \omega(1).$$

Example codes

① Tillich-Zémor[09] hypergraph product codes

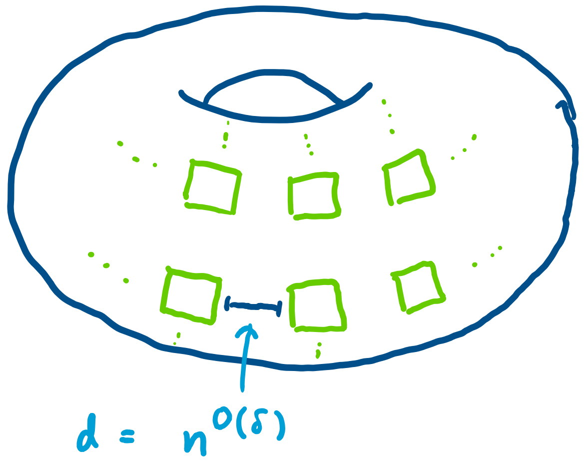


$k = \Theta(n)$

$d = \Theta(\sqrt{n})$

Possibly full NLTS.

② Punctured toric code with $\Omega(n^{1-\delta})$ holes.



$k = \Theta(n^{1-\delta})$

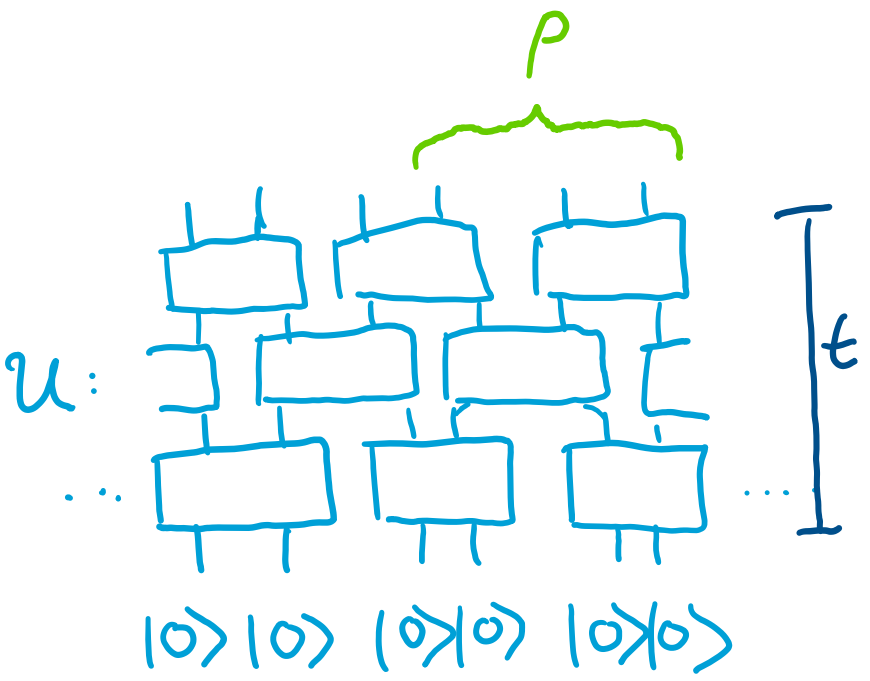
$d = \Theta(n^{\delta})$

for $tr(H\rho) \le O(n^{1-2\delta})$, $cc(\rho) \ge \Omega(\delta \log n)$.

Not full NLTS since 2D.

Part II : Defs and Key Lemmas

Circuit Complexity



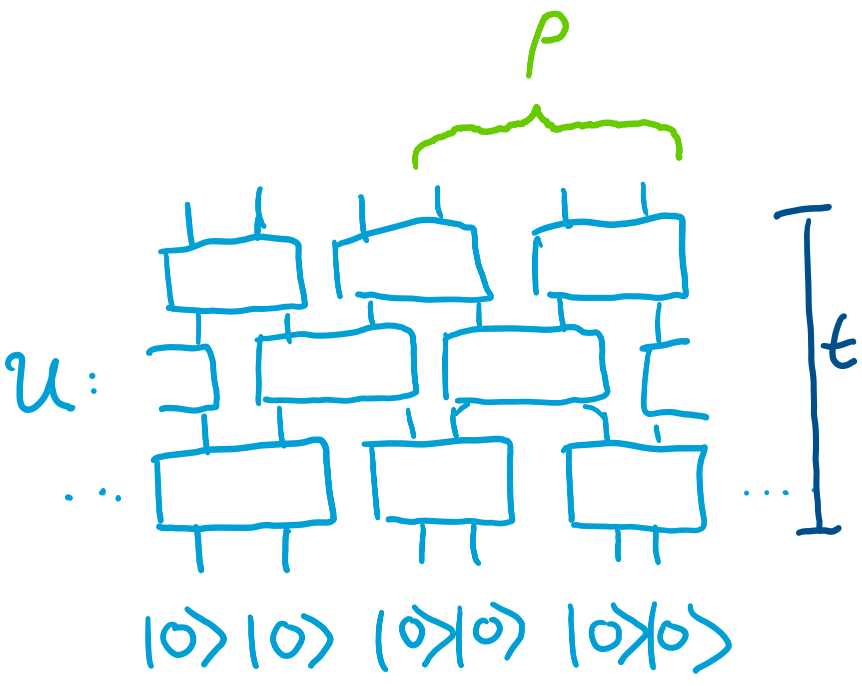
$cc(\rho)$ = min depth t of any ckt exactly producing ρ.

Fact A state has $cc \le 1$ iff it is a tensor product state.

Fact Given a $O(1)$-local Ham. H and a state ρ of $cc(\rho) = t$, there is a classical alg. for computing $\mathrm{tr}(H\rho)$ (i.e. energy) in time $\mathrm{poly}(n) \cdot \exp(\exp(t))$.

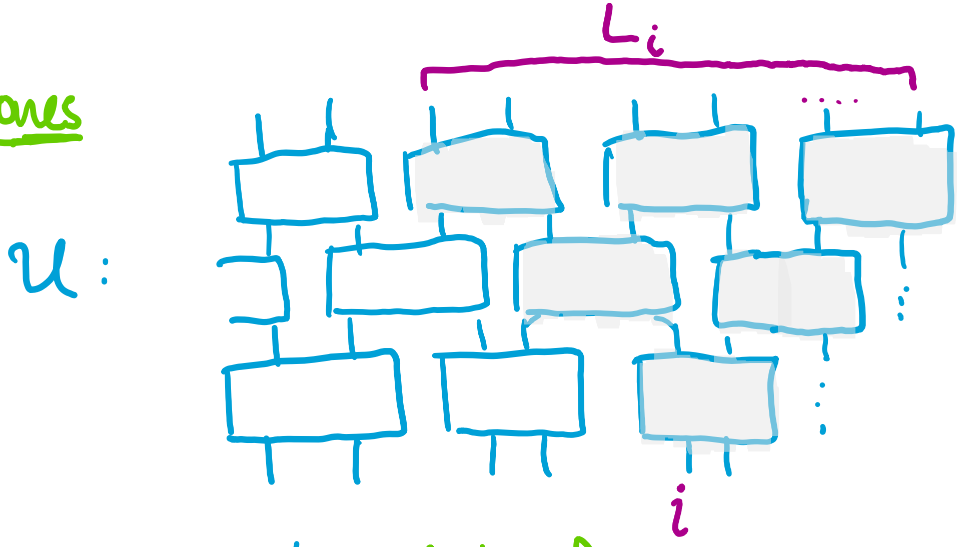
Pf. Each term $\mathrm{tr}(H_i \rho)$ depends on only the reduced computation on $O(2^t)$ qubits.

Circuit Complexity



$cc(\rho)$ = min depth t of any ckt exactly producing ρ.

Lightcones



Fact 1 $|L_i| < 2^t$ ← depth of ckt.

reduced density matrix on L_i.

Fact 2 $tr_{-i}(\mathcal{U}\rho\,\mathcal{U}^\dagger) = tr_{-i}(\mathcal{U}\rho_{L_i}\mathcal{U}^\dagger)$

Pf: next page.

Circuit Complexity

reduced density matrix on L_i.

Fact 2 $\mathrm{tr}_{-i}(U\rho U^\dagger) = \mathrm{tr}_{-i}(U\rho_{L_i}U^\dagger)$

Pf: Let A be any operator on qubit i.

$$\mathrm{tr}(A\,\mathrm{tr}_{-i}(U\rho U^\dagger))$$

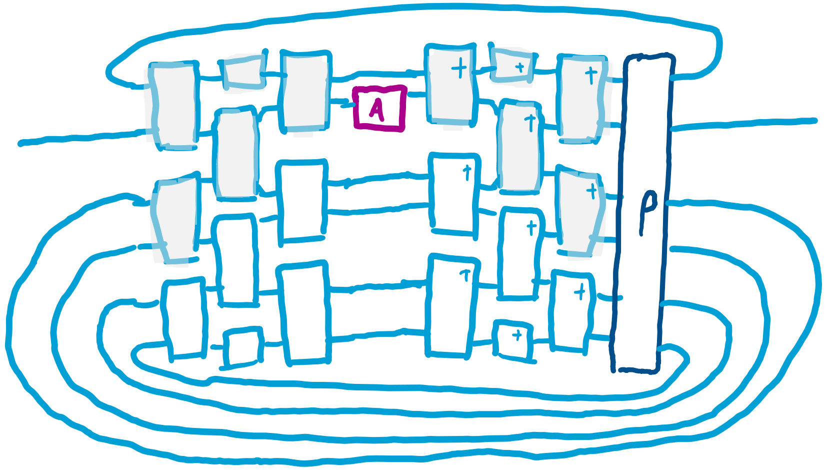
$$= \mathrm{tr}(U^\dagger A U \rho)$$

$$= \mathrm{tr}(U^\dagger A U \rho_{L_i})$$

$$= \mathrm{tr}(A\,\mathrm{tr}_{-i}(U\rho_{L_i}U^\dagger))$$

$$\mathrm{tr}(U^\dagger A U \rho)$$

$=$



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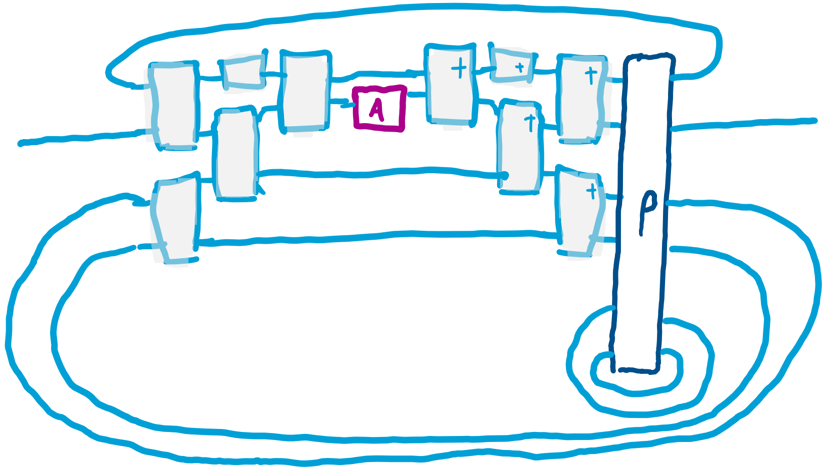


$$= \mathrm{tr}(U^\dagger A U \rho_{L_i})$$

QPCP $\Rightarrow$ NLTS (modulo NP $\neq$ QMA)

QPCP: given Local Ham H is min energy $= 0$ or $\geq \frac{1}{10} n$ is QMA-hard.

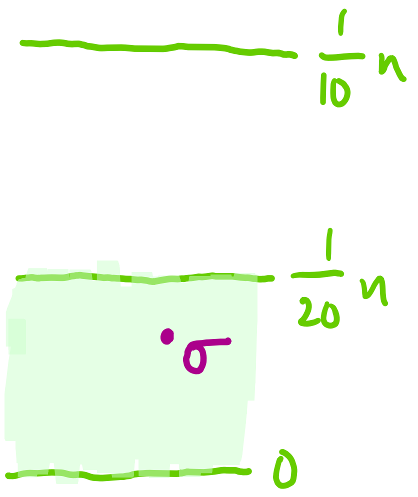
+

$\neg$NLTS: For every $\varepsilon > 0$, every Local Ham H, $\exists$ a state of cc $= O_\varepsilon(1)$ of energy $\leq \varepsilon n$.

$\Rightarrow$

QMA pf that min energy $= 0$ is a state ρ s.t. $\mathrm{tr}(H\rho) = 0$.

Instead, by $\neg$NLTS, $\exists$ a state σ with cc $= O(1)$ & $\mathrm{tr}(H\sigma) \leq \frac{1}{20} n$. Let $\mathcal{U}$ be defining ckt.



Claim: $\mathcal{U}$'s description is a classical witness for problem.

min energy $= 0 \Rightarrow$ classically check $\mathrm{tr}(H\sigma) \leq \frac{1}{20} n$

min energy $\geq \frac{1}{10} n \Rightarrow \forall\, \mathcal{U}$, $\mathrm{tr}(H\sigma) \geq \frac{1}{10} n$.

because depth $\mathcal{U}$ is $O(1)$.

$\Rightarrow$ QMA $=$ NP.

Error-correcting Codes

i.e. Local Indistinguishability.

Knill-Laflamme conditions:

Can correct an error E iff

$$\Pi E \underbrace{\Pi}_{\text{projector on the codespace.}} = \eta_E \Pi$$

Code has dist d, if it can correct all errors of size $< d$.

Let S be a set of qubits of $|S| < d$. (correctable region)

Then $\forall$ codestates ρ, $\rho_S = tr_{-S}(\rho)$ is an invariant.

Pf: E = any operator acting only on S.

$$tr(E\rho) = tr(E\Pi\rho\Pi)$$

$$= tr(\Pi E \Pi \rho)$$

$$= tr(\eta_E \Pi \rho)$$

$$= \eta_E. \leftarrow \rho \text{ independent.}$$

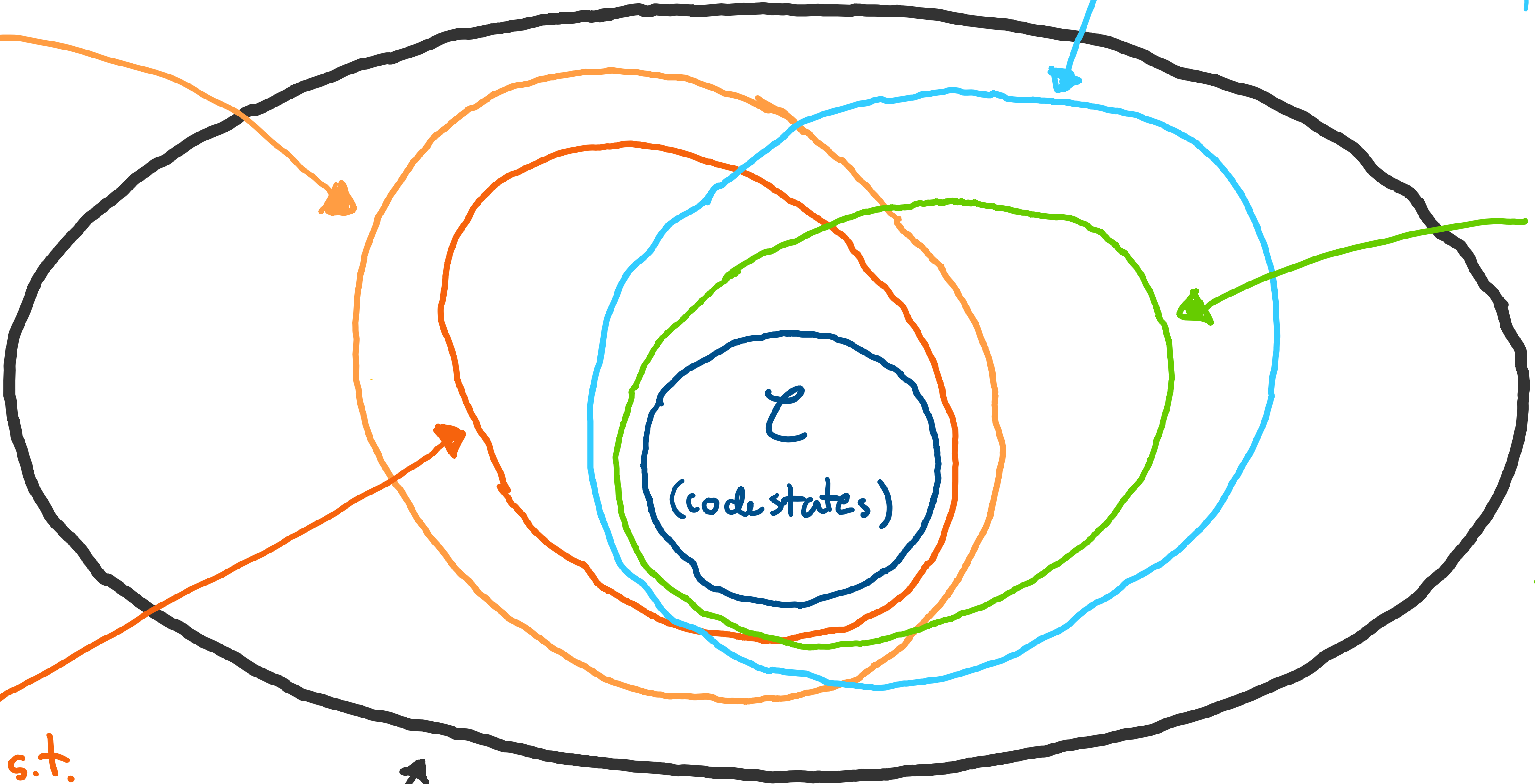
■

Part II : Sketch of proof techniques

The set of low-energy states

Combinatorial states: $\Pr_i\left(\mathrm{tr}(H_i\rho) \neq 0\right) \leq \varepsilon.$

ε-smooth states: $\forall i$, $\mathrm{tr}(H_i\rho) \leq \varepsilon.$

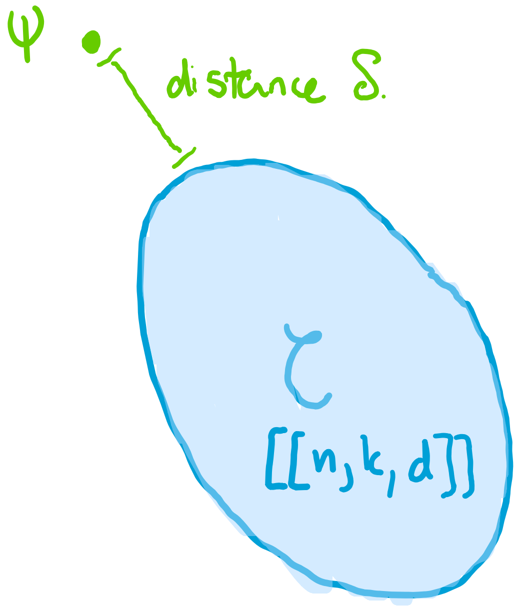


low-error states: take codestate ψ and change up to $O(\varepsilon n)$ qubits

ε-dist states: $\exists$ codestate ψ s.t. $\|\rho - \psi\|_1 \leq \varepsilon.$

all low energy states: $\mathrm{tr}(H\rho) \leq \varepsilon n$

Warmup: Circuit LBs for low-distance states



Let ψ be a state dist δ from $\mathcal{C}$. What is $cc(\psi)$?

Folklore: For any codestate ρ, $cc(\rho) = \Omega(\log d)$.

For simplicity, let's only consider pure states and circuits without ancillas.

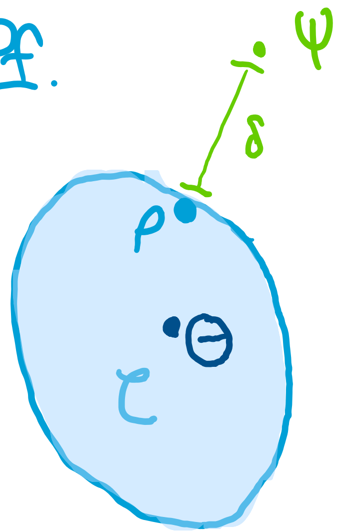
<u>Thm</u>: Let $\sqrt{\delta} < k/n$, for any state $|\psi\rangle$ of dist δ from $\mathcal{C}$.

$$\Rightarrow cc(\psi) \geq \Omega(\log d).$$

Warmup: Circuit LBs for low-distance states

Thm: Let $\sqrt{\delta} < k/n$, for any state $|\psi\rangle$ of dist δ from $\mathcal{C}$. $\Rightarrow cc(\psi) \geq \Omega(\log d)$.

Pf. ρ be closest codestate to ψ. Θ be encoded maximally mixed state.



① Let R be a region of $|R| < d$.

$$\psi_R \underbrace{\approx_\delta}_{\text{distance}} \rho_R \underbrace{=}_{\text{local indistinguishability}} \Theta_R.$$

② Let $|\psi\rangle = \mathcal{U}|0\rangle^{\otimes n}$ for $\mathcal{U}$ of depth t s.t. $2^t < d$.

$$|0\rangle\langle 0| = \mathrm{tr}_{-i}(\mathcal{U}^\dagger \psi \mathcal{U}).$$

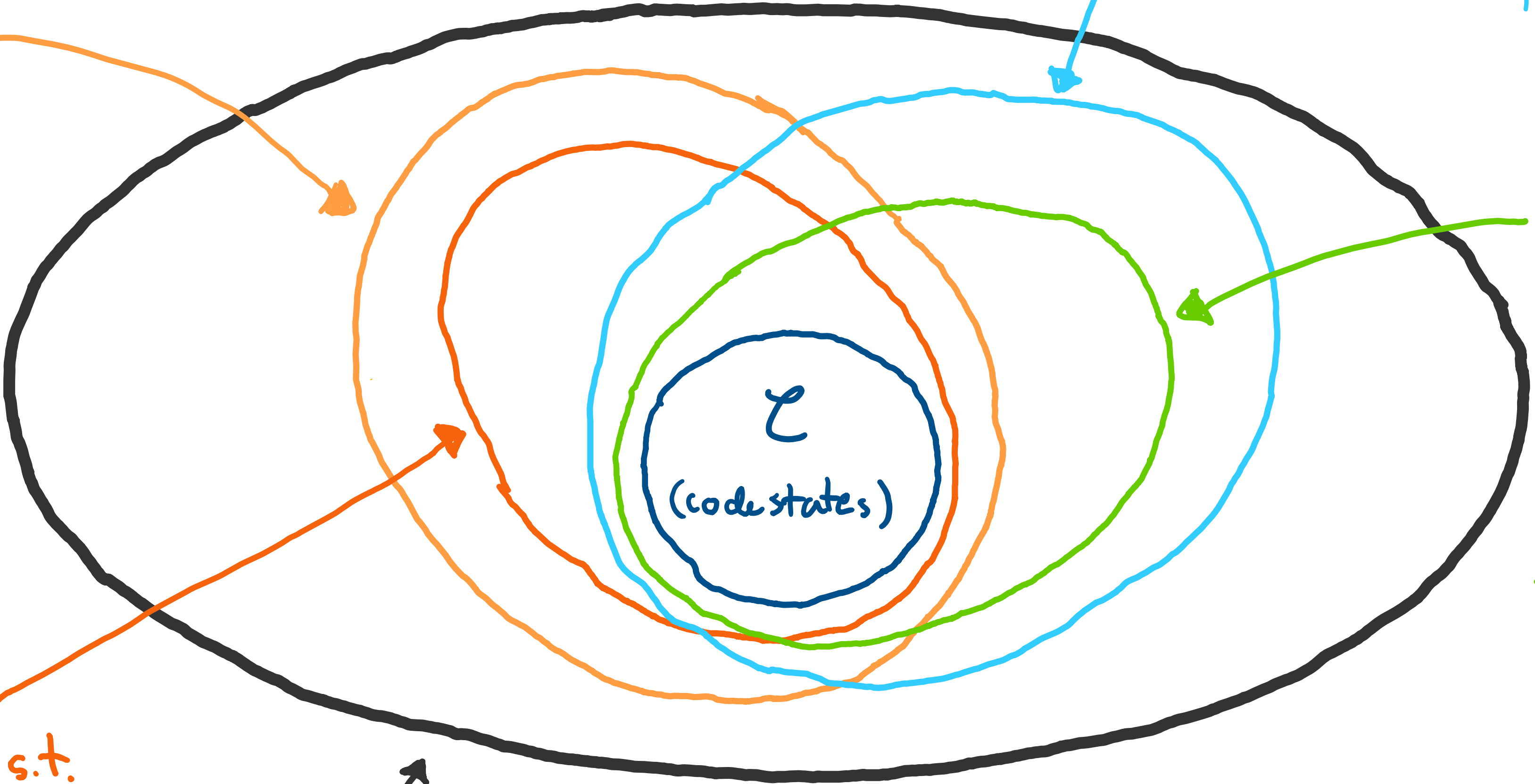
③ $|0\rangle\langle 0| = \mathrm{tr}_{-i}(\mathcal{U}^\dagger \psi \mathcal{U}) = \mathrm{tr}_{-i}(\mathcal{U}^\dagger \psi_{L_i} \mathcal{U}) \underset{①}{\approx_\delta} \mathrm{tr}_{-i}(\mathcal{U}^\dagger \Theta_{L_i} \mathcal{U}) = \mathrm{tr}_{-i}(\mathcal{U}^\dagger \Theta \mathcal{U})$. [green ='s from the lightcone argument]

④ $S(\mathrm{tr}_{-i}(\mathcal{U}^\dagger \Theta \mathcal{U})) \leq \sqrt{\delta}$. ⑤ $k = S(\Theta) = S(\mathcal{U}^\dagger \Theta \mathcal{U}) \leq \sum_{i=1}^{n} S(\mathrm{tr}_{-i}(\mathcal{U}^\dagger \Theta \mathcal{U})) \leq \sqrt{\delta}\, n$. $\perp$.

The set of low-energy states

Combinatorial states:
$\Pr_i(\operatorname{tr}(H_i \rho) \neq 0) \leq \varepsilon$.

ε-smooth states: $\forall i$, $\operatorname{tr}(H_i \rho) \leq \varepsilon$.

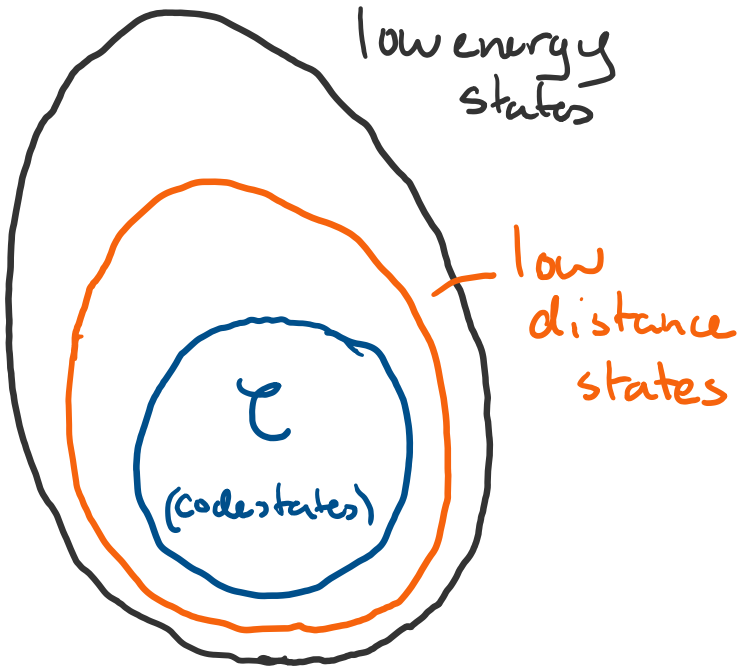


low-error states: take codestate ψ and change up to $O(\varepsilon n)$ qubits

ε-dist states: $\exists$ codestate ψ s.t. $\|\rho - \psi\|_1 \leq \varepsilon$.

all low energy states: $\operatorname{tr}(H\rho) \leq \varepsilon n$

Extending the argument to low-energy states



If $k = \Omega(n)$, then circuit LBs for all low-distance states.

LDPC Stabilizer Codes

All checks are tensor products of a few Paulis.

$$\mathcal{C} = \{ |\psi\rangle : C_i |\psi\rangle = |\psi\rangle \ \forall i \}.$$

$$D_s = \{ |\psi\rangle : C_i |\psi\rangle = (-1)^{s_i} |\psi\rangle \ \forall i \} \text{ for } s \in \{0,1\}^n.$$

Rmk : Local indistinguishability holds for each eigenspace D_s : Region R s.t. $|R| < d$. Then, ρ_R invariant over each D_s. (but can depend on s).

Main Thm

Let $\mathcal{C}$ be a $[[n,k,d]]$ stabilizer LDPC code and ϕ a n qubit mixed state s.t. $\mathrm{tr}(H_{\mathcal{C}}\phi) \le \varepsilon n$. Then,

$$cc(\phi) \ge \Omega\left(\min\left\{\log d,\ \log\left(\frac{k}{n}\cdot \underbrace{\frac{1}{\varepsilon \log(1/\varepsilon)}}_{\ge \frac{1}{\sqrt{\varepsilon}}}\right)\right\}\right).$$

Sketch of low-energy argument : gentle measurement

Let ϕ be a n-qubit mixed state and $\mathcal{U}$ a depth t ckt on m qubits constructing ϕ.

Let $\varepsilon_i = \mathrm{tr}(H_i \phi)$ and $\sum_{i=1}^{N} \varepsilon_i \le \varepsilon n$.

Wlog can assume $m \le n \cdot 2^t$.

Let ψ be ϕ after coherently measuring each stabilizer into $N = O(n)$ extra ancilla.

And Ψ be ϕ after incoherently measuring.

Ψ has a constructing ckt W of depth $t + O(1)$ ← due to LDPC.

① Let R be a region of the qubits.

$$F(\psi_R, \Psi_R) \ge 1 - \sum_{\text{syndrome} \ni \text{qubit } i \in R} \varepsilon_i .$$

Pf: Roughly gentle measurements from commuting measurements.

Sketch of low-energy argument : introducing entropy

Ψ = incoherently measured ϕ, the low-depth low-energy state.

$$\mathcal{E}(\rho) := \frac{1}{4^k} \sum_{x,z \in \{0,1\}^k} (\bar{X}^x \bar{Z}^z)(\rho)(\bar{X}^x \bar{Z}^z)^\dagger$$ i.e. logical completely decohering channel

Define $\Theta = \mathcal{E}(\Psi)$. $\Rightarrow$ $S(\Theta) \geq k$.

(2) Let R be a region of qubits s.t. $|R| < d$.

Then $\Psi_R = \Theta_R$.

Pf: Local indistinguishability per eigenspace D_s. Both Ψ and Θ are CQ states with same dist.

Sketch of low-energy argument: Putting it together.

ψ = coherently measured ϕ. $\quad \Theta$ = logically completely decohered Ψ.

Ψ = incoherently measured ϕ.

① $F(\psi_R, \Psi_R) \geq 1 - \sum_{i \in R} \varepsilon_i$ (syndrome measurement)

② $\Psi_R = \Theta_R$ when $|R| < d$.

③ For any qubit j,

$\mathrm{tr}_{-j}(W^\dagger \psi W) = |0\rangle\langle 0|$.

④ Assuming $2^{t+O(1)} < d$, then

$F(\mathrm{tr}_{-j}(W^\dagger \psi W), \mathrm{tr}_{-j}(W^\dagger \Theta W))$

$\geq F(\mathrm{tr}_{-j}(W^\dagger \psi_{L_j} W), \mathrm{tr}_{-j}(W^\dagger \Theta_{L_j} W))$

$\geq 1 - \sum \varepsilon_i$. $\longleftarrow$ by ① + ②.

Sketch of low-energy argument : Bounding the rate

④ $F\left(|0\rangle\langle 0|, \mathrm{tr}_{-j}\left(W^\dagger \Theta W\right)\right) \geq 1 - \sum_{i \in L_j} \varepsilon_i := 1 - \varepsilon_{L_j}$

$\mathbb{E}\ \varepsilon_{L_j} = O(2^t \varepsilon)$.

⑤ $S\left(\mathrm{tr}_{-j}\left(W^\dagger \Theta W\right)\right) \leq \varepsilon_{L_j} \log\left(\frac{1}{\varepsilon_{L_j}}\right)$.

⑥ $k \leq S(\Theta) = S\left(W^\dagger \Theta W\right)$

$\leq \sum_j S\left(\mathrm{tr}_{-j}\left(W^\dagger \Theta W\right)\right)$

$\vdots \qquad \vdots$

$\leq O\left[\left(m + O(n)\right)\left(2^{t+O(1)} \varepsilon \log(1/\varepsilon)\right)\right]$

$= O\left(2^{2t} n \varepsilon \log(1/\varepsilon)\right)$.

$\therefore$ if Φ has depth t and energy $\leq \varepsilon n$,

$t \geq \Omega\left(\min\left\{\underbrace{\log d}_{\text{assumed for calculating fidelity.}}, \underbrace{\log\left(\frac{k}{n} \cdot \frac{1}{\varepsilon \log 1/\varepsilon}\right)}_{\text{due to bound on the rate.}}\right\}\right)$.

Argument Recap

Thm $\mathcal{C} = [[n,k,d]]$ LDPC Stab code. ϕ is a depth t state of energy $\leq \varepsilon n$.

$$\Rightarrow \quad t = \Omega\left(\min\left\{ \log d, \log\left(\frac{k}{n} \cdot \frac{1}{\varepsilon \log 1/\varepsilon}\right)\right\}\right).$$

① Since LDPC stabilizer, can measure all terms in constant depth.

② Using local indistinguishability, show that (on average) $\Theta_R \approx \Psi_R$ for Ψ the coherently measured state and Θ, a high entropy state.

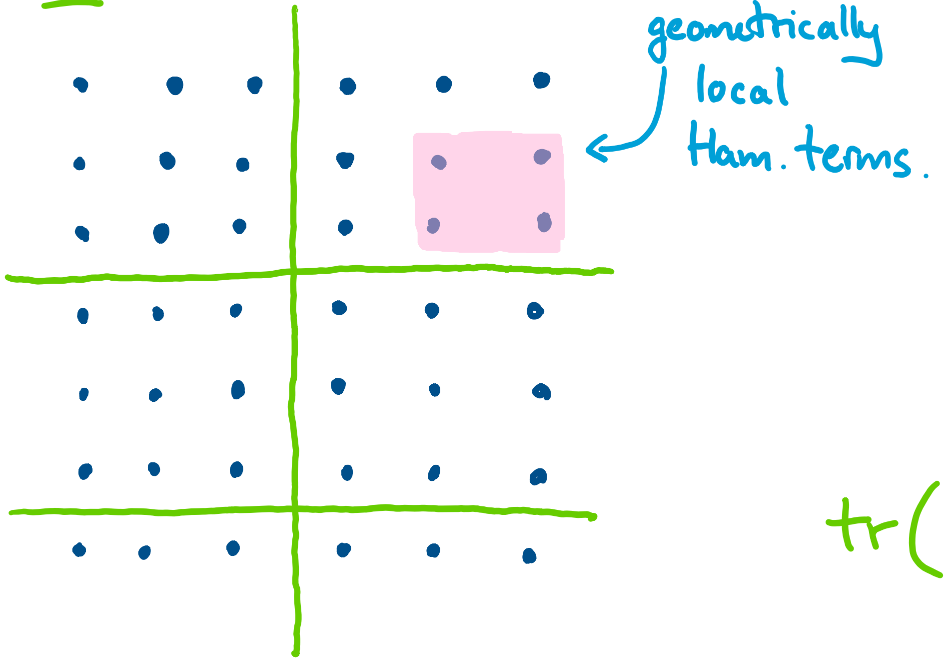
③ Use the low-depth of ϕ to bound the entropy of Θ, yielding the conclusion.

Next Steps

This argument holds for the punctured toric code, a 2D code.

Full NLTS cannot hold for Hamiltonians on constant dimensional lattice.

Pf:



Break space into $O(1/\varepsilon)$ sized regions $R_1, \ldots, R_{O(\varepsilon n)}$.

Let ψ be a ground state.

$$\tilde{\psi} = \bigotimes_{i=1}^{O(\varepsilon n)} \psi_{R_i} \leftarrow cc(\tilde{\psi}) \leq O(2^{1/\varepsilon}).$$

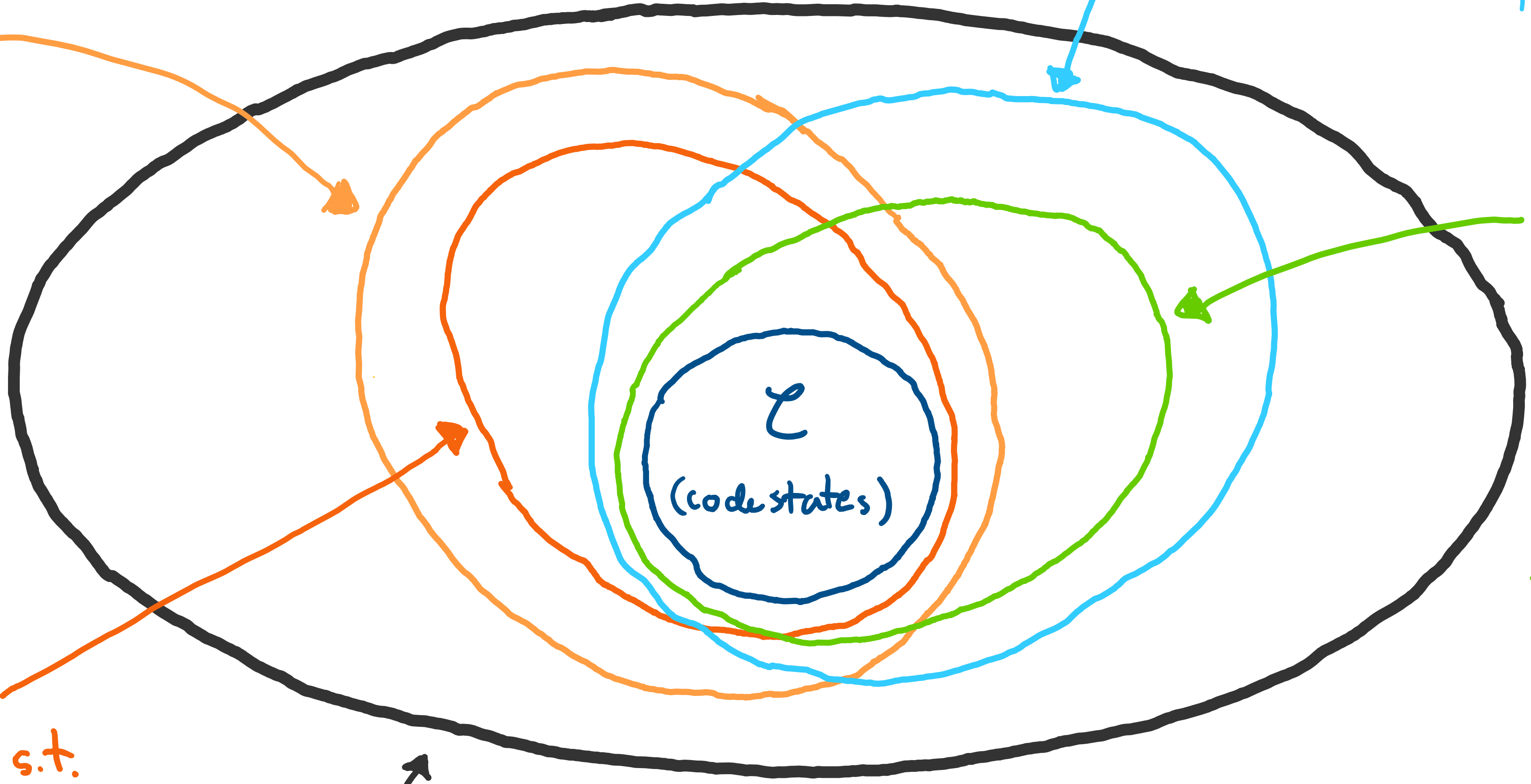
$\mathrm{tr}(H\tilde{\psi}) \leq \varepsilon n$ as only few Ham. terms are ignored.

The set of low-energy states

Combinatorial states: $\Pr_i\left(\mathrm{tr}(H_i \rho) \neq 0\right) \leq \varepsilon$.

ε-smooth states: $\forall i$, $\mathrm{tr}(H_i \rho) \leq \varepsilon$.

low-error states: take codestate ψ and change up to $O(\varepsilon n)$ qubits



ε-dist states: $\exists$ codestate ψ s.t. $\|\rho - \psi\|_1 \leq \varepsilon$.

all low energy states: $\mathrm{tr}(H\rho) \leq \varepsilon n$

CNLTS Conjecture [Eldar, Harrow'17]

This cutting argument holds for weakening of NLTS conjecture.

<u>Combinatorial NLTS</u> : $\exists$ $\varepsilon > 0$, and family of Ham. $H^{(n)}$ s.t.
for any H' defined by removing εn terms from $H^{(n)}$,
the cc of any groundstate of H' is superconstant.

CNLTS is also open! And a barrier as our almost linear NLTS thm is true for non-CNLTS Hamiltonians.

Other open questions

Is the local Hamiltonian problem with promise gap $\frac{1}{n^{0.01}}$ also QMA-hard?

Our almost NLTS result is the corresponding result.

Circuit lower bounds for other "approximations" of the groundspace have been proved.

Such as the NLETS results [Eldar, Harrow'17; **N**, Vazirani, Yuen'18] for low-error states.

Can these results be reproduced with these new techniques?

Thank you & any questions?

Paper available at arXiv: 2011.02044

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