Good approximate QLDPC codes from spacetime Hamiltonians

Chinmay Nirkhe

joint work with

Thom Bohdanowicz
Caltech

Elizabeth Crosson
University of New Mexico

Henry Yuen
University of Toronto

arXiv:1811.00277

QIP 2019, STOC 2019

Berkeley
UNIVERSITY OF CALIFORNIA

nirkhe@cs.berkeley.edu

Yesterday…

Henry Yuen talked about how

clock Hamiltonians + error-correcting codes $\Rightarrow$ low-error states

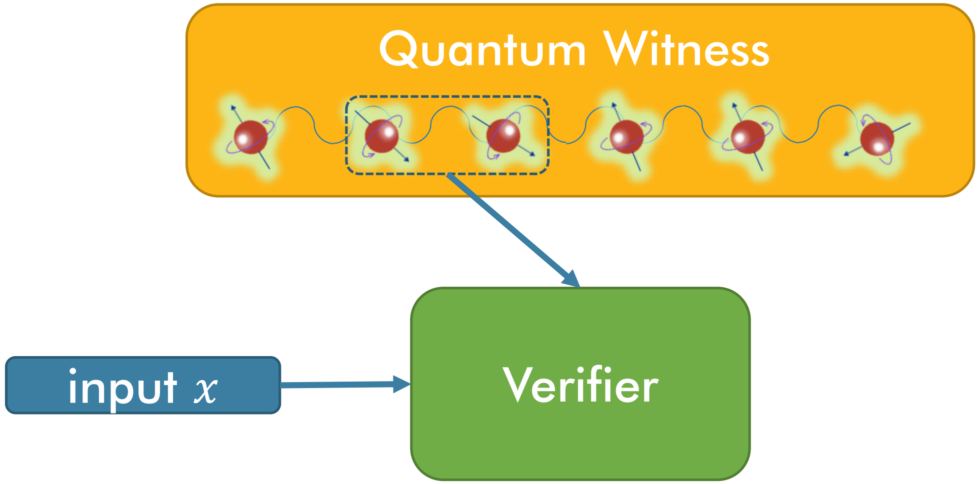
and began talking about approximate error-correction

This talk will be a (self-contained) follow-up that elaborates on these constructions.

Why study error-correcting codes?

Quantum fault tolerance

- [Gottesman[09]] QLDPC ⇒ fault tolerance quantum computation with constant overhead

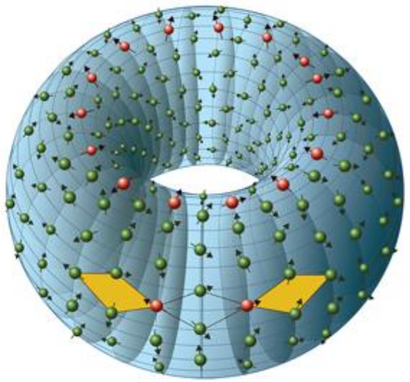


Quantum PCP conjecture

- Hardness of approximation in quantum setting
- Entanglement at room temperature

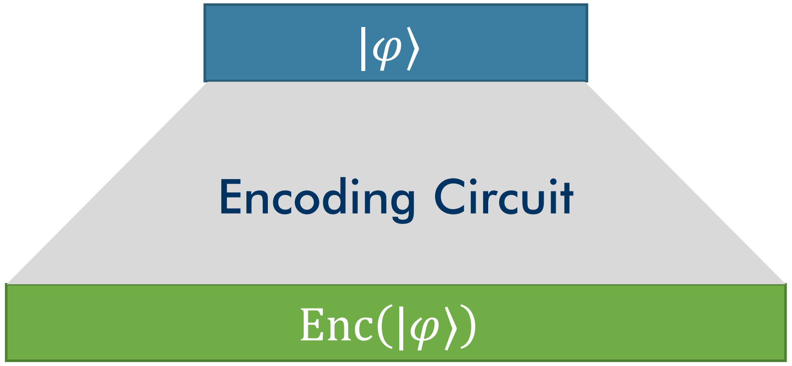
Interesting local Hamiltonians

- with robust entanglement properties
- toric code, color codes, etc.

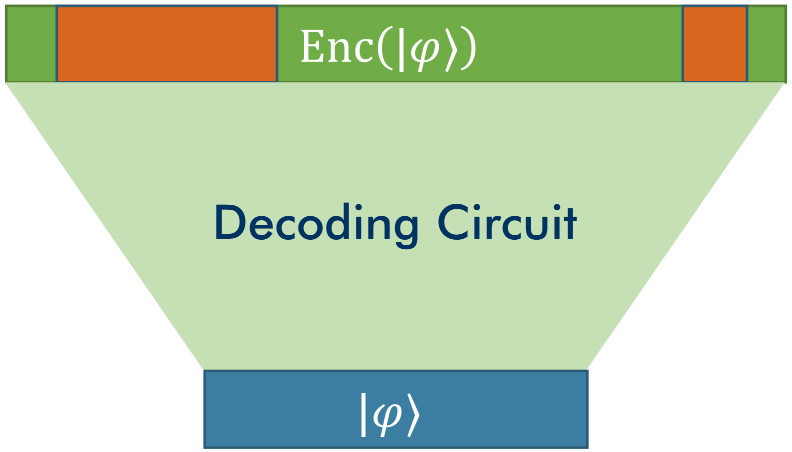


What properties are we looking for in a code?

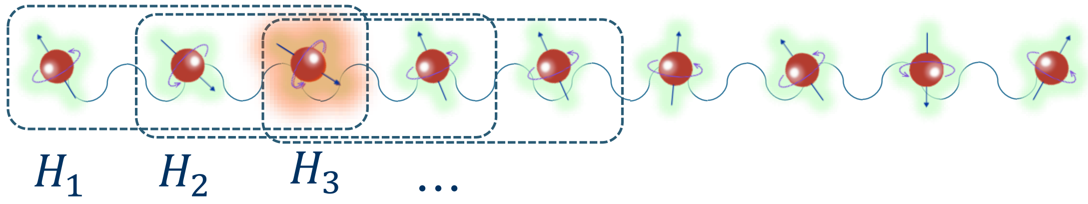
Rate



Distance



Stabilizer weight (locality)



Rate: $\frac{k}{n} = \Omega(1)$

Distance: $d = \Omega(n)$

Locality: $O(1)$

We show that optimal rate, distance and locality parameters *are* possible (modulo polylog corrections)

if we go beyond stabilizer codes to

non-commuting and approximate codes


Quantum error correcting codes
ex. toric code
Local Hamiltonians
(with robust entanglement)
this talk

Outline

- Coding theory definitions
- Uniformization via sorting circuits
- Spacetime Hamiltonians
- Spectral gap analysis

What is a LDPC code?

Classically, a code $\mathcal{C}$ is a dim k subspace of $\mathbb{Z}_2^n$.

A linear code can be defined by a matrix $H \in \mathbb{Z}_2^{n\times(n-k)}$.

$$\mathcal{C} = \{x \in \mathbb{Z}_2^n : Hx = 0\}$$

L ow

D ensity

P arity

C heck

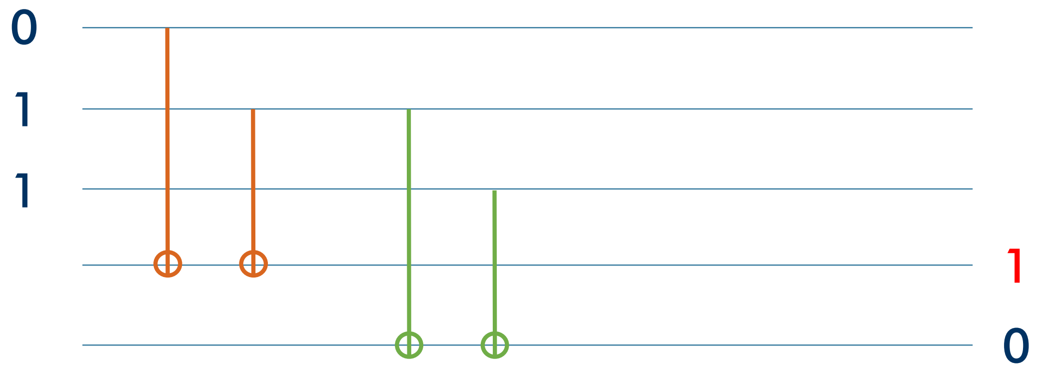
$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 = x_2 = x_3 \quad \Rightarrow \quad \mathcal{C} = \{000, 111\}$$

H has c-locality if H is c-row sparse and c-column sparse.

Benefits of an LDPC code

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$



Since the checks overlap, they can't be parallelized and must be done in series.

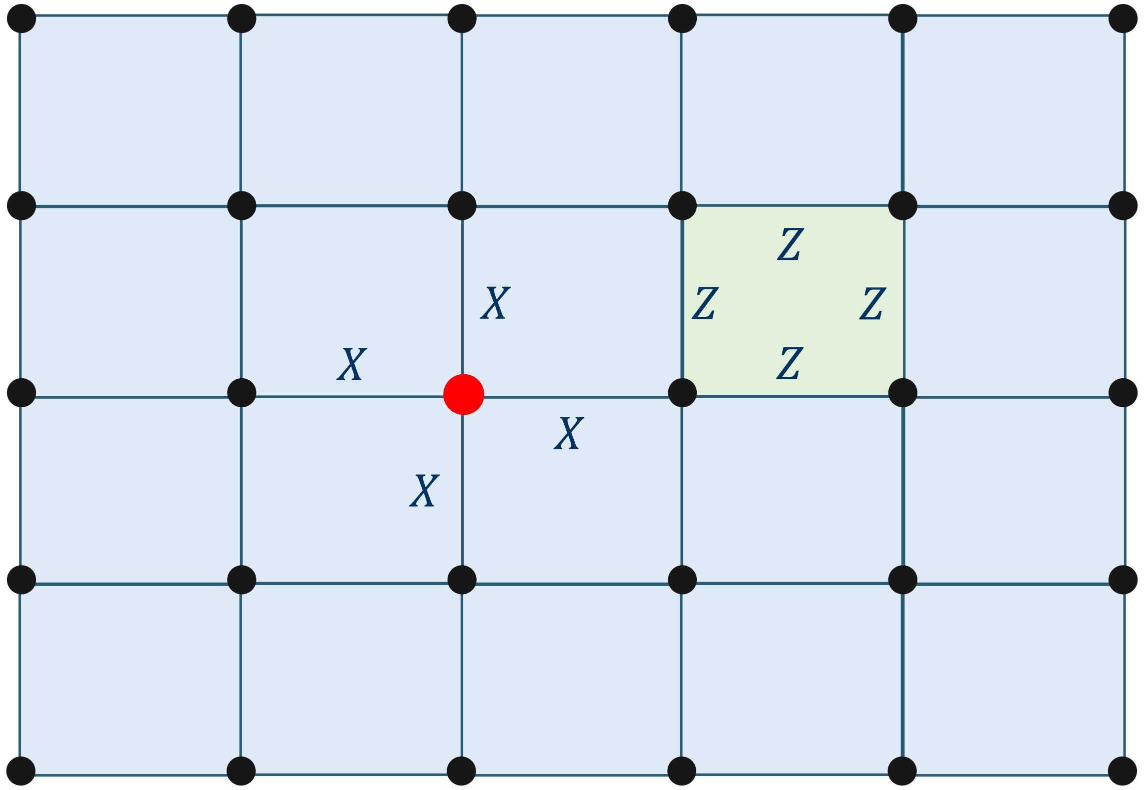
If the code is c-local, then the checks can be parallelized into $c^3 + c$ depth circuit.

Proof: Each check shares bits with at most c^2 other checks. By coloring argument, requires $c^2 + 1$ rounds. Each round requires depth c.

Quantum LDPC codes

For CSS codes (codes that handle X errors and Z errors separately), definition is easy…

both parity check matrices H_X and H_Z need to have low density.



ex. toric code

locality: 4

rate: $2/n$

distance: $O(\sqrt{n})$

Can we do better?

Best known stabilizer codes

- [Tillich-Zemor[13]]
  - rate: $\Omega(1)$
  - distance: $O(\sqrt{n})$
  - locality: $O(1)$
- [Freedman-Meyer-Luo[02]]
  - rate: $\Omega(1/n)$
  - distance: $O(\sqrt{n \log n})$
  - locality: $O(1)$
- [Bravyi-Hastings[13]]
  - rate: $\Omega(1)$
  - distance: $O(n)$
  - locality: $O(\sqrt{n})$

To do better, we probably need to go past stabilizer codes!

Going past stabilizer codes

Let $H_1, H_2, \dots, H_m$ be a set of c-local projectors acting on n qubits. ← not necessarily commuting

Define the code-space $\mathcal{C}$ as the mutual eigenspace:

$$\mathcal{C} = \left\{|\varphi\rangle \in (\mathbb{C}^2)^{\otimes n} \middle| \langle\varphi|H_i|\varphi\rangle = 0 \ \forall\ H_i\right\}$$

$H = H_1 + \cdots + H_m$ is c-QLDPC if additionally each qubit participates in at most c terms H_i.

First attempt [N-Vazirani-Yuen[18]]

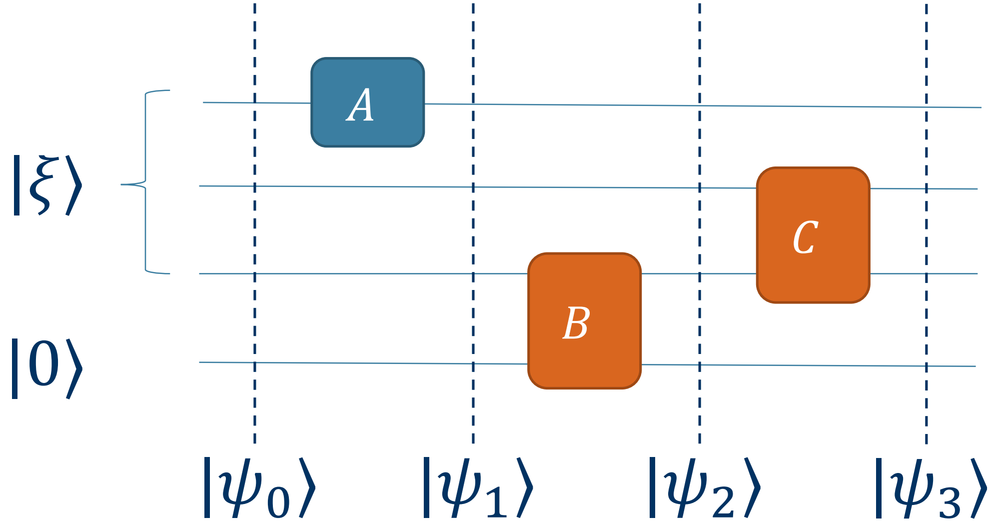
CSS codes exist with linear rate and distance, but lack locality.

Create a Hamiltonian whose ground-space is almost exactly that of a CSS code but is locally checkable.

First attempt [N-Vazirani-Yuen[18]]

Create a Hamiltonian whose ground-space is almost exactly that of a CSS code but is locally checkable.

Express a computation as the ground-state of a 5-local Hamiltonian (Feynman-Kitaev clock Hamiltonian) [Kitaev[99]]



$$|\psi_0\rangle = |\xi\rangle|0\rangle$$
$$|\psi_1\rangle = A|\psi_0\rangle$$
$$|\psi_2\rangle = B|\psi_1\rangle$$
$$|\psi_3\rangle = C|\psi_2\rangle$$

Together, $\{|\psi_t\rangle\}$ are a "proof" that the circuit was executed correctly. But, $|\widetilde{\Psi}\rangle = |\psi_0\rangle|\psi_1\rangle \dots |\psi_T\rangle$ is not locally-checkable.

Instead, the following "clock" state* is:

$$|\Psi\rangle = \frac{1}{\sqrt{T+1}}\sum_{t=0}^{T}|t\rangle|\psi_t\rangle$$

*Quantum analog of Cook[71]-Levin[73] Tableau.

First attempt [N-Vazirani-Yuen[18]]

Create a Hamiltonian whose ground-space is almost exactly that of a CSS code but is locally checkable.

Let $C = C_T C_{T-1} \dots C_1$ be a circuit with gates $\{C_i\}$ and let $|\psi_0\rangle = |\xi\rangle|0\rangle^{\otimes n-k}$ be an initial state for $|\xi\rangle \in (\mathbb{C}^2)^{\otimes k}$.

There is a local Hamiltonian with ground space of:

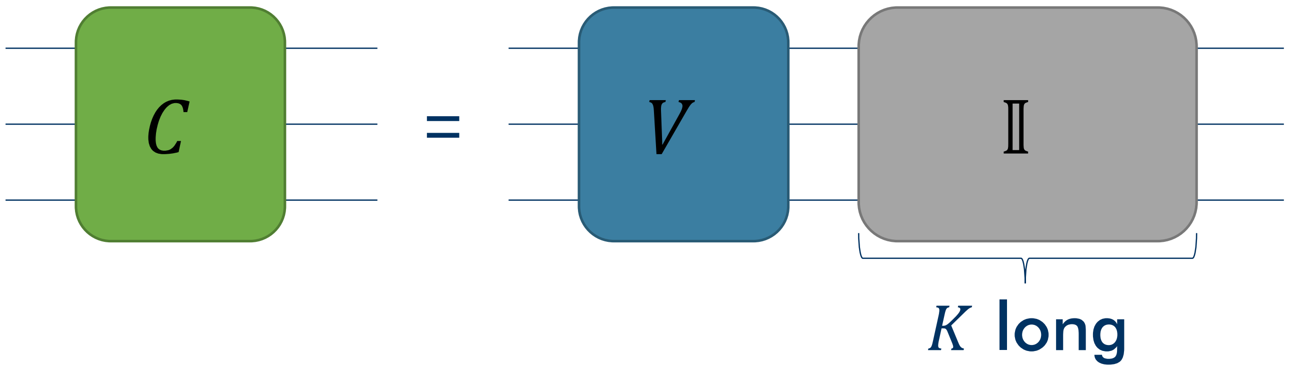
$$\mathcal{G} = \left\{ |\Psi_\xi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\text{unary}(t)\rangle \otimes |\psi_t\rangle : \begin{array}{c} |\psi_t\rangle = C_t|\psi_{t-1}\rangle, \\ |\psi_0\rangle = |\xi\rangle|0\rangle^{\otimes(n-k)} \end{array} \right\}.$$

First attempt [N-Vazirani-Yuen[18]]

Create a Hamiltonian whose ground-space is almost exactly that of a CSS code but is locally checkable.

Let V be the encoding circuit for a good CSS code.

Choose $K = O(T_V \delta^{-2})$.



Construct the clock Hamiltonian for this “padded” circuit C.

First attempt [N-Vazirani-Yuen[18]]

The groundspace of H is $\approx$ the groundspace of a CSS code tensored with junk.

$$\mathcal{G}_C = \left\{ \frac{1}{\sqrt{T_C + 1}} \sum_{t=0}^{T} |t\rangle|\psi_t\rangle \ : \ \begin{matrix} |\psi_t\rangle = C_t C_{t-1} \dots C_1 |\psi_0\rangle, \\ |\psi_0\rangle = |\xi\rangle|0\rangle^{\otimes(n-k)} \end{matrix} \right\}$$

But for $t \geq T_V$, $|\psi_t\rangle = V|\psi_0\rangle$. Thus, $1 - O(\delta^{-2})$ fraction of $|\psi_t\rangle = V|\psi_0\rangle$.

$$\mathcal{G}_C \approx \frac{1}{\sqrt{T_C + 1}} \sum_{t=0}^{T} |t\rangle \ \otimes \left\{ V|\psi_0\rangle : |\psi_0\rangle = |\xi\rangle|0\rangle^{\otimes(n-k)} \right\}.$$

Plus, $\mathcal{G}_C$ is the ground-space of a 5-local Hamiltonian!

However, some qubits participate in many terms H_t.

First attempt [N-Vazirani-Yuen[18]]

However, some qubits participate in many terms H_t.

H_t checks that the slice $|t\rangle|\psi_t\rangle$ and the slice $|t+1\rangle|\psi_{t+1}\rangle$ satisfy

$$|\psi_{t+1}\rangle = U_t|\psi_t\rangle$$

t^{th} gate of circuit

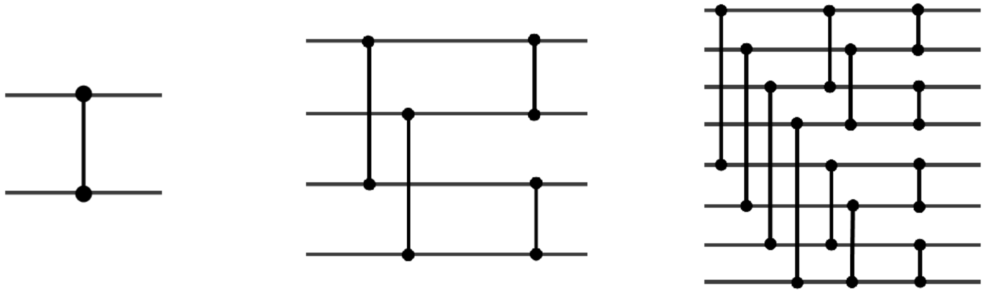
Locality of the code corresponds to the connectivity of the qubits in the circuit.

Minimize connectivity of the qubits in the circuit.

Localizing the circuit via bitonic sorting circuits

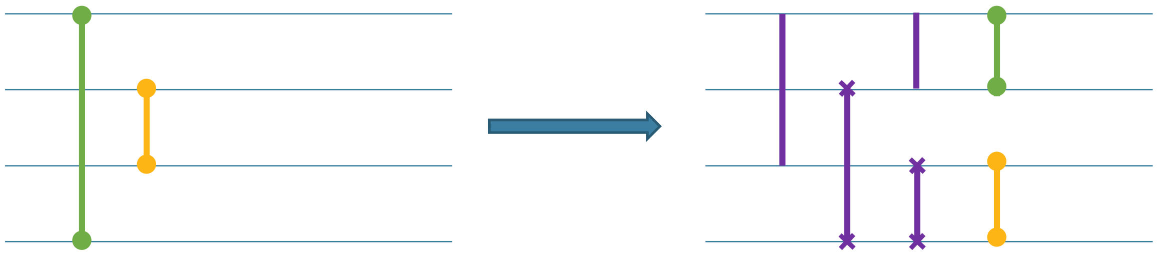
Minimize connectivity of the qubits in the circuit.

Theorem [Batcher65]: There is a circuit of depth $\log^2 n$ with $\log n$ connectivity sorting n elements.



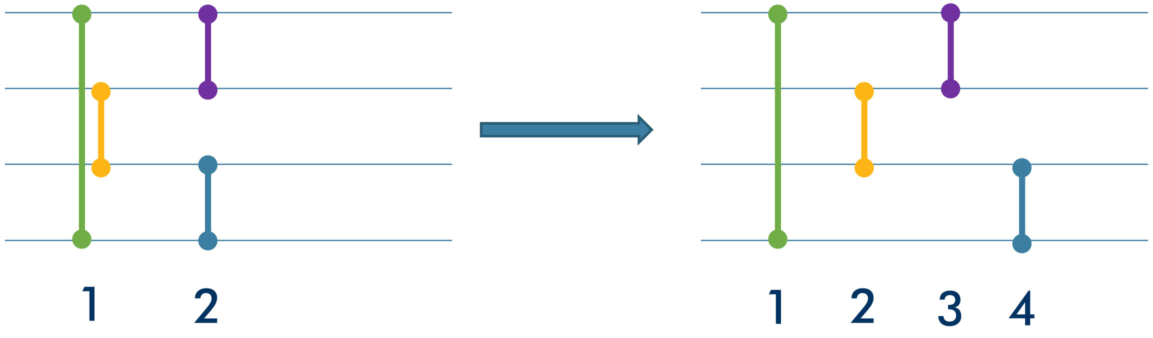
(a) (b) (c)

Can stretch circuit by $\log^2 n$ mult. depth and reduce connectivity to $\log n$.



Can be used anywhere to simplify circuit connectivity in any situation.

Long clocks and brittle Hamiltonians



For Feynman-Kitaev clock Hamiltonian each layer of circuit needs exactly 1 gate.

This yields long clocks and brittle Hamiltonians.

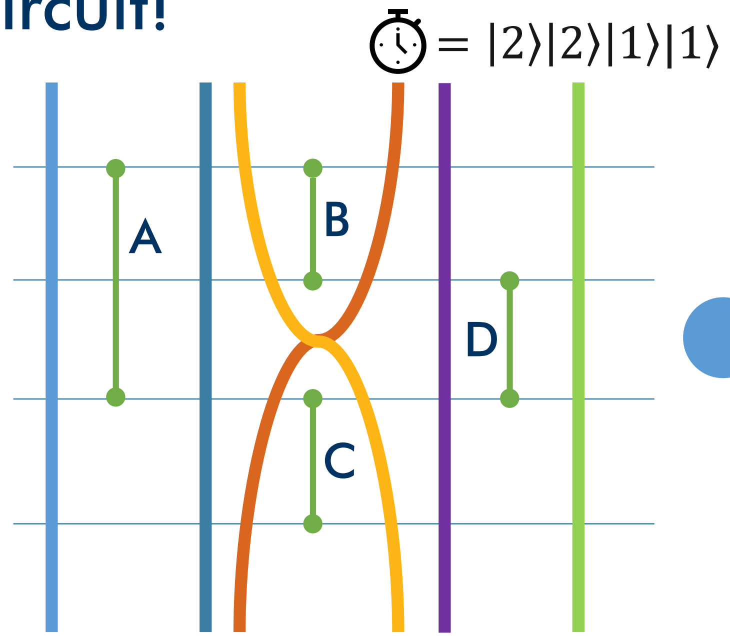
Brittle Hamiltonian: Small spectral gap.

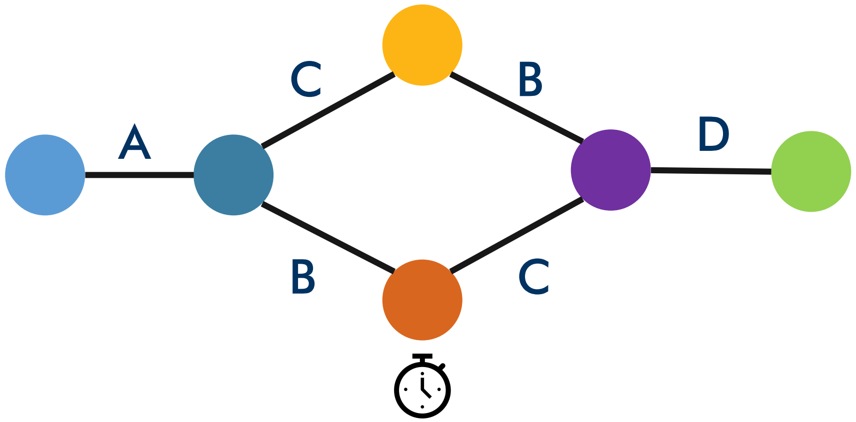
Not satisfying any 1equation $|\psi_{t+1}\rangle = U_t|\psi_t\rangle$ has energy $O(1/|C|)$ with $|C|$ the number of gates.

Long clocks and brittle Hamiltonians

This yields long clocks and brittle Hamiltonians.

There are more than $|C|$ partial computations of a circuit!





Build Hamiltonian with ground-state of uniform superposition overall partial computations τ:

$$\sum |\tau\rangle|\psi_\tau\rangle$$

Space-time Hamiltonian [Breuckmann-Terhal[14]]

Long clocks and brittle Hamiltonians

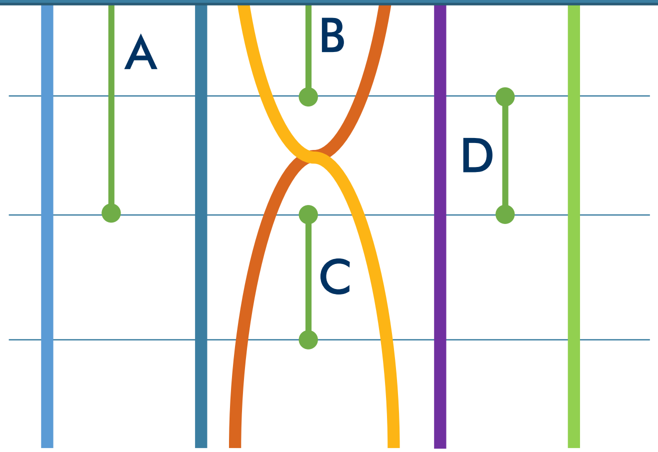
Theorem: This yields a Hamiltonian for whom the spectral gap scales

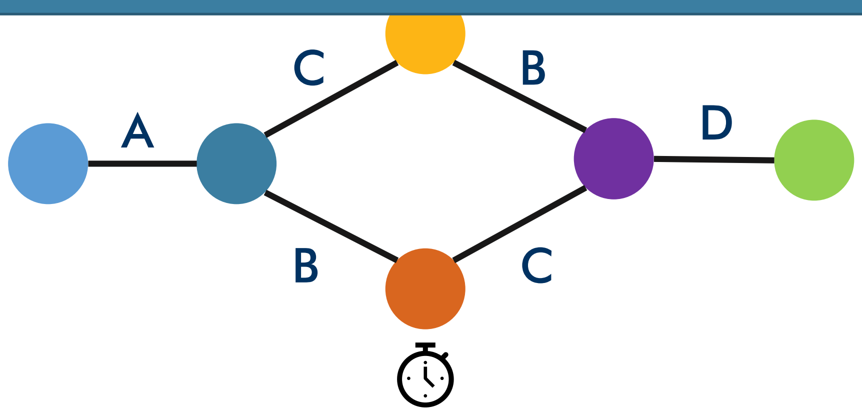
$$\widetilde{\Omega}\left(\frac{1}{n^{3.09}\mathrm{depth}(C)^2}\right)$$

Instead of $\Omega\left(\frac{1}{|C|^2}\right)$ as in standard Feynman-Kitaev clock Hamiltonian

n with
of
sition
al
τ:

$$\sum |\tau\rangle|\psi_\tau\rangle$$





Space-time Hamiltonian [Breuckmann-Terhal[14]]

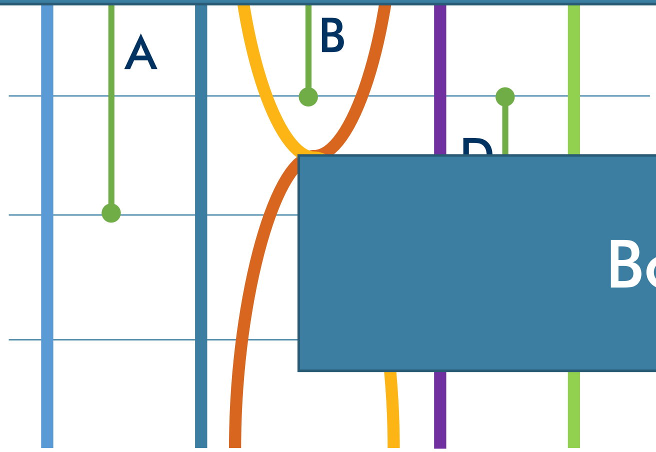
Long clocks and brittle Hamiltonians

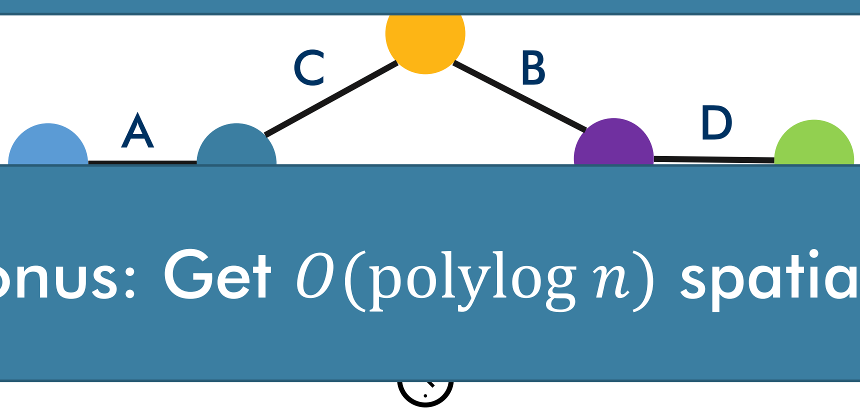
Theorem: This yields a Hamiltonian for whom the spectral gap scales

$$\widetilde{\Omega}\left(\frac{1}{n^{3.09}\mathrm{depth}(C)^2}\right)$$

Instead of $\Omega\left(\frac{1}{|C|}\right)$ as in standard Feynman-Kitaev clock Hamiltonian

n with
of
sition
al
τ:





$$\sum |\tau\rangle|\psi_\tau\rangle$$

Bonus: Get $O(\mathrm{polylog}\, n)$ spatial locality

[Breuckmann-Terhal[14]]

Approximate decoding

$$|\Psi\rangle = \sum_{\tau} |\tau\rangle|\psi_t\rangle \approx \sum_{\tau} |\tau\rangle \otimes |\psi_{final}\rangle$$

over all valid partial computations

due to padding with identity gates

$$\mathcal{E}(|\Psi\rangle) \approx \mathcal{E}_1\left(\sum_{\tau} |\tau\rangle\right) \otimes \mathcal{E}_2(|\psi_{final}\rangle)$$

Approximate decoding:

1. Trace out clock registers
2. Apply underlying code decoding procedure

Low-complexity error-detection

This is achievable to the spacetime construction!

With prob. $1 - 2^{-\Omega(\log^2 n)}$, can detect Pauli errors with $\mathrm{polylog}\, n$ depth circuits even when the # of errors is bigger than the distance.

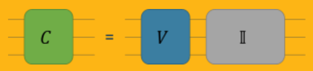
Because of the approximate nature, does not imply can detect all errors.

Not a property of [N-Vazirani-Yuen[18]] code.

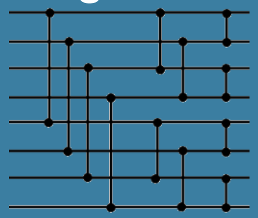
Construction recap

A code with linear rate and distance and $O(\log^3 n)$ depth encoding circuit [Brown-Fawzi[13]]

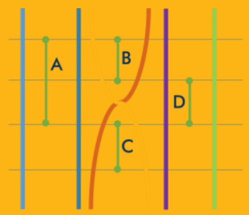
Pad the circuit with identity gates

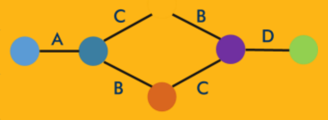


Uniformize the connectivity of the circuit using bitonic sorting circuits



Build spacetime Hamiltonian of resulting code [Breuckmann-Terhal[14]]





Spectral gap analysis

In the traditional clock Hamiltonian,

$$H_t = \frac{1}{2}(|t+1\rangle \otimes U - |t\rangle \otimes I)(\langle t+1| \otimes U^\dagger - \langle t| \otimes U^\dagger)$$

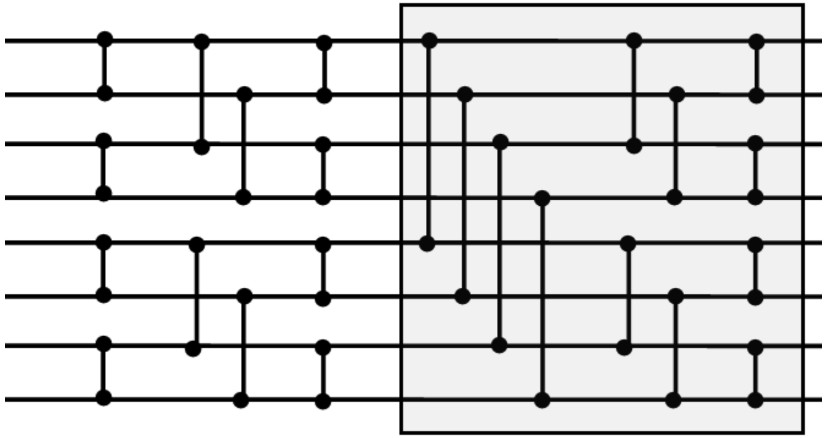
Analyzing the spectral gap of $\sum H_t$ is equivalent to analyzing the spectral gap of this Markov chain:

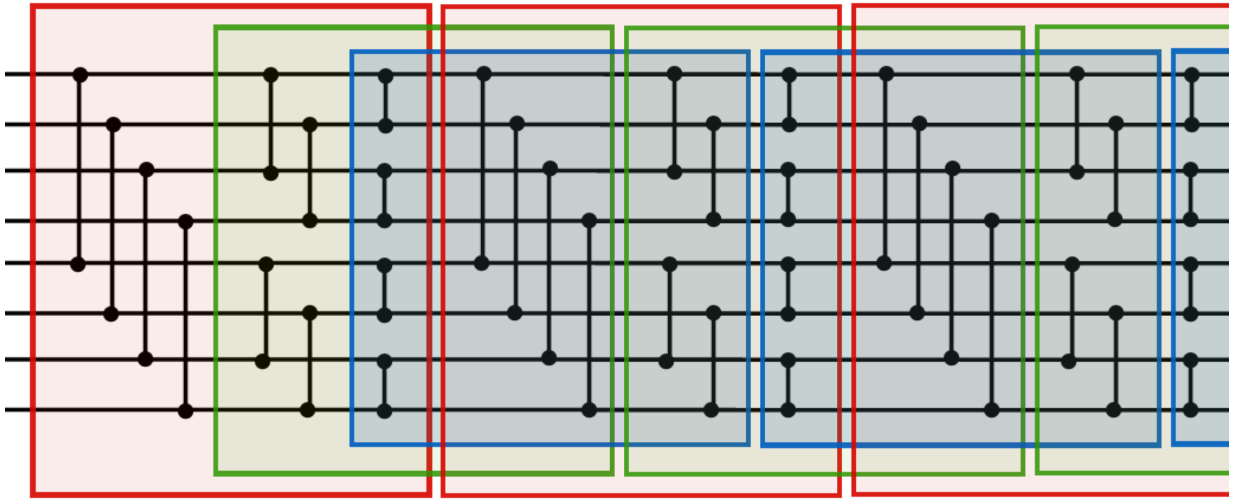
$$\lambda_{\min} = \Omega\left(\frac{1}{n^2}\right)$$

Spectral gap analysis

Def: minimum non-zero eigenvalue of Hamiltonian H

Map the Hamiltonian to a Markov chain over the space of valid partial computations

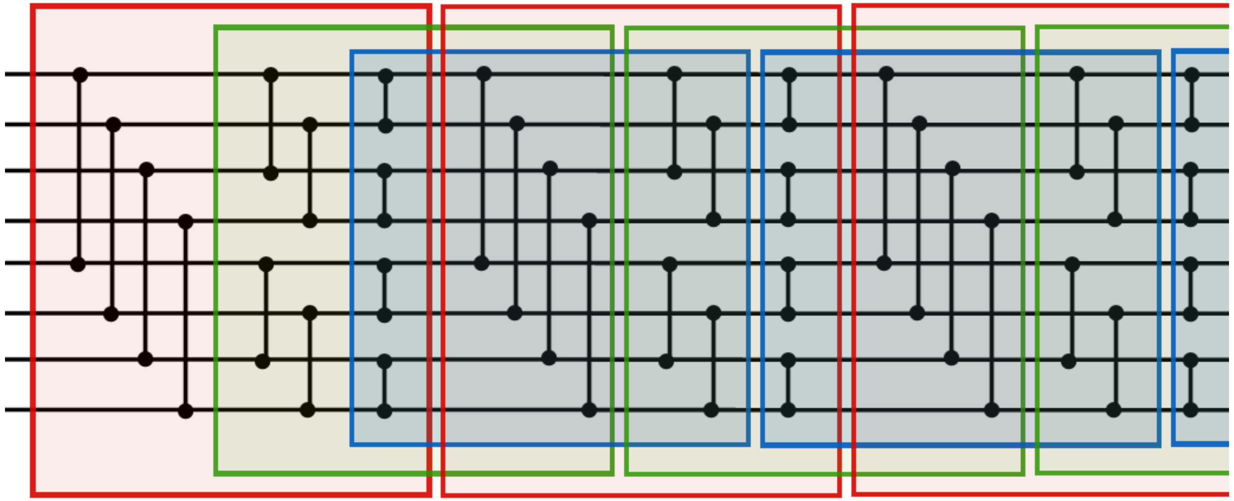




Spectral gap analysis

Spectral gap of the code is based on the mixing time of valid configurations of a bitonic block

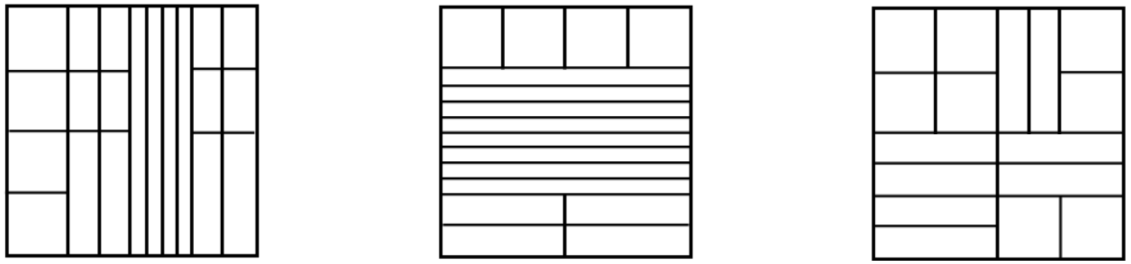
True of all constructions built from bitonic sorting circuits



Spectral gap analysis

Spectral gap of the code is based on the mixing time of valid configurations of a bitonic block

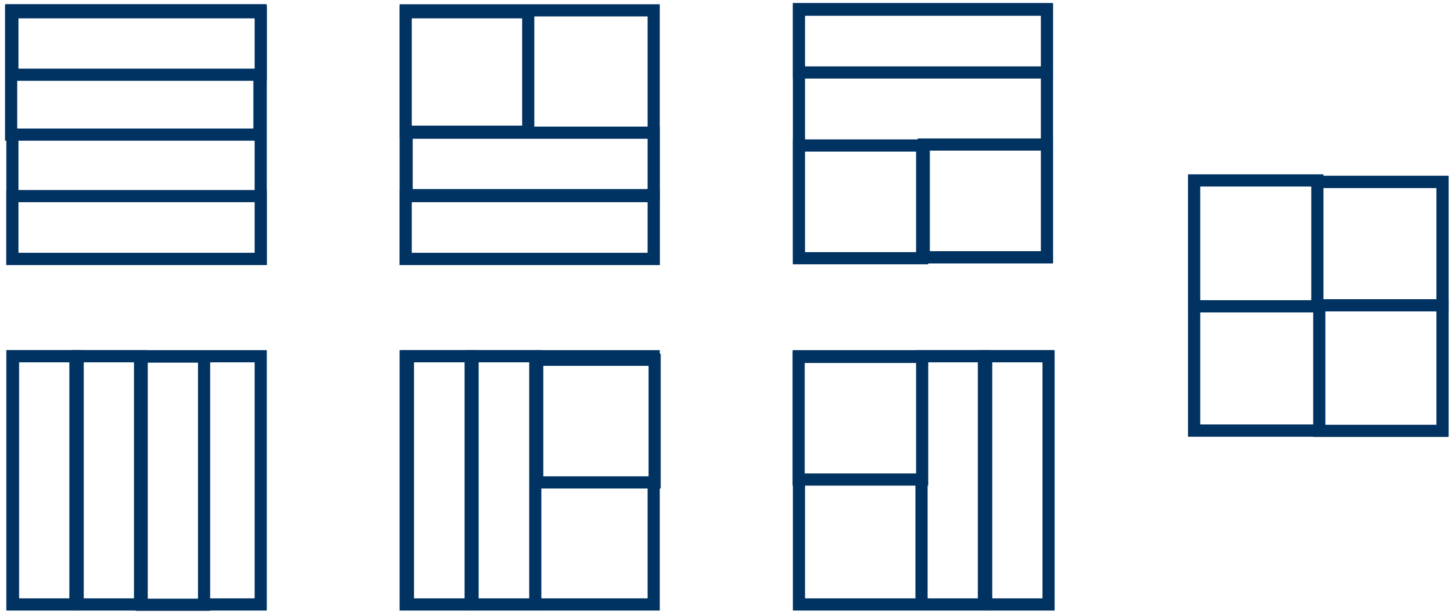
We noticed that bitonic blocks look similar to a structure called dyadic tilings studied in [Cannon-Levin-Stauffer[17]]



Dyadic tilings are ways of covering the unit square by 2^d rectangles with corner coordinates at multiples of 2^{-d}

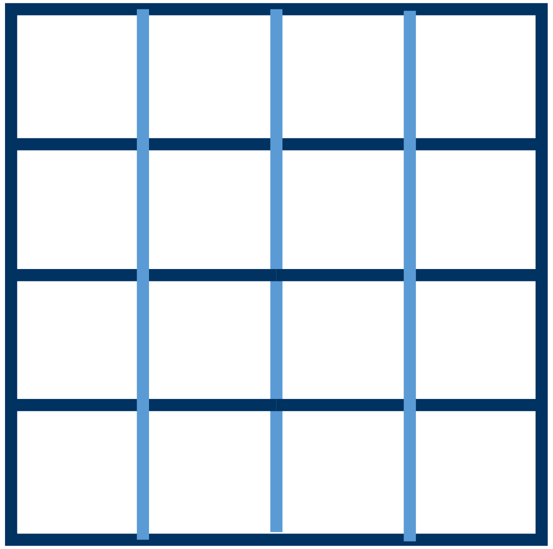
Spectral gap analysis

Spectral gap of the code is based on the mixing time of valid configurations of a bitonic block



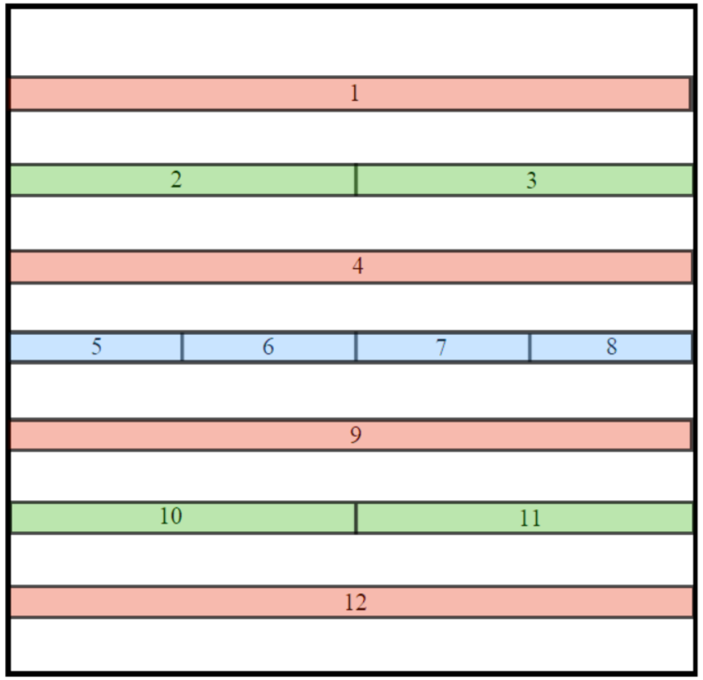
Spectral gap analysis

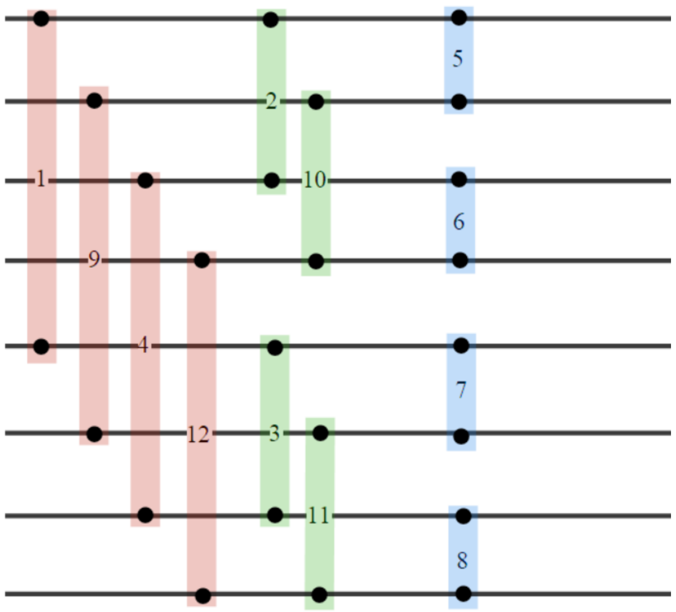
Spectral gap of the code is based on the mixing time of valid configurations of a bitonic block



Spectral gap analysis

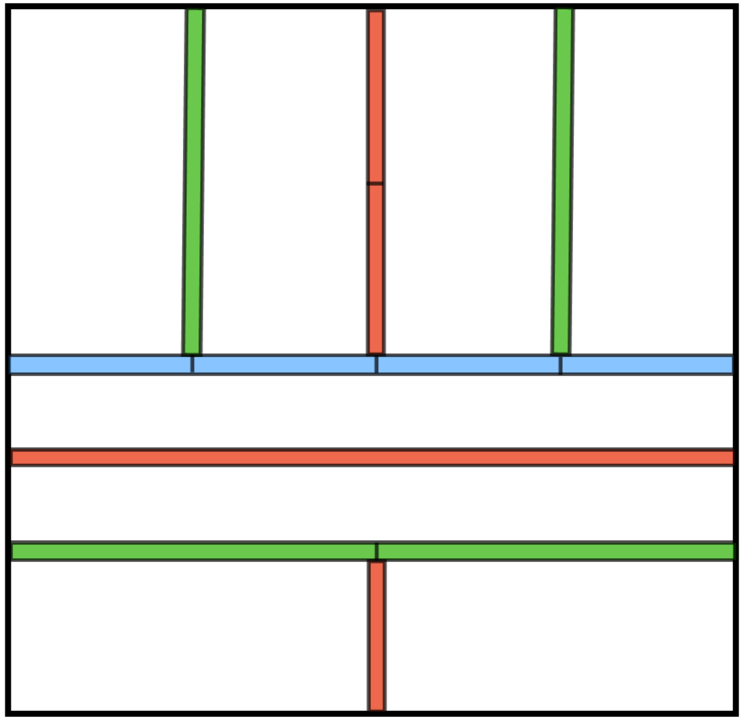
Spectral gap of the code is based on the mixing time of valid configurations of a bitonic block

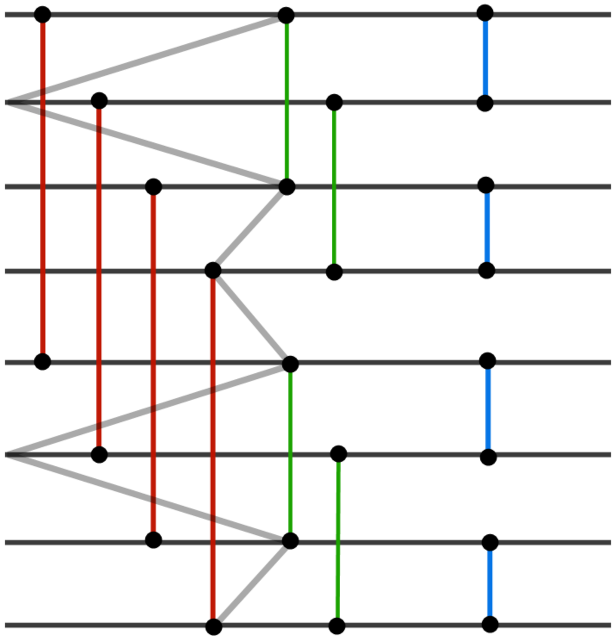




Spectral gap analysis

Spectral gap of the code is based on the mixing time of valid configurations of a bitonic block



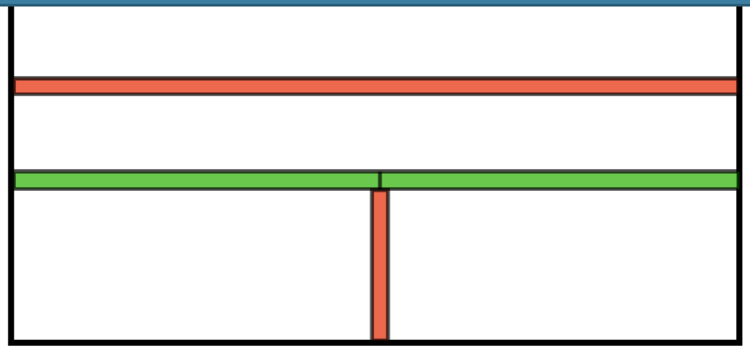


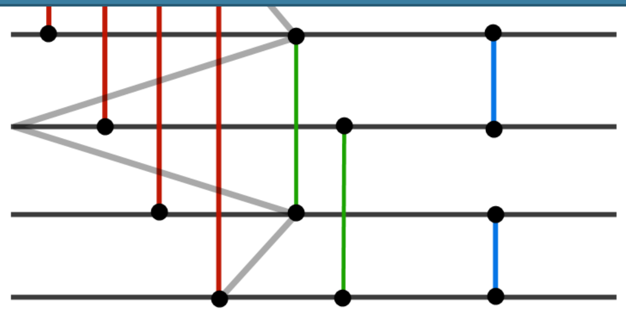
Spectral gap analysis

Spectral gap of the code is based on the mixing time of valid configurations of a bitonic block

Theorem: The spectral gap of this Hamiltonian is

$$\widetilde{\Omega}\left(n^{-3.09}\right).$$





Summary of results

We constructed a new type of code based on spacetime Hamiltonians.

It has the following properties:

- rate: $\Omega(\frac{1}{\operatorname{polylog} n})$
- distance: $\Omega(\frac{n}{\operatorname{polylog} n})$
- spatial-locality: $\Omega(\operatorname{polylog} n)$
- spectral-gap: $\Omega(n^{-3.09})$

Along the way, we also learned about

- localizing large stabilizers using circuit-to-Hamiltonian constructions
- uniformizing circuits with bitonic sorting networks
- analysis of uniform circuits via Markov chain techniques

What does this teach us?

First, this isn't the ''perfect'' error-correcting code or is realistic

Relaxing the requirements of stabilizer codes is helpful

- Code-space as the ground-space of a sum of non-commuting projectors
- Approximate error-correction

There are connections between computation and error-correction that we don't fully understand!

Thanks!