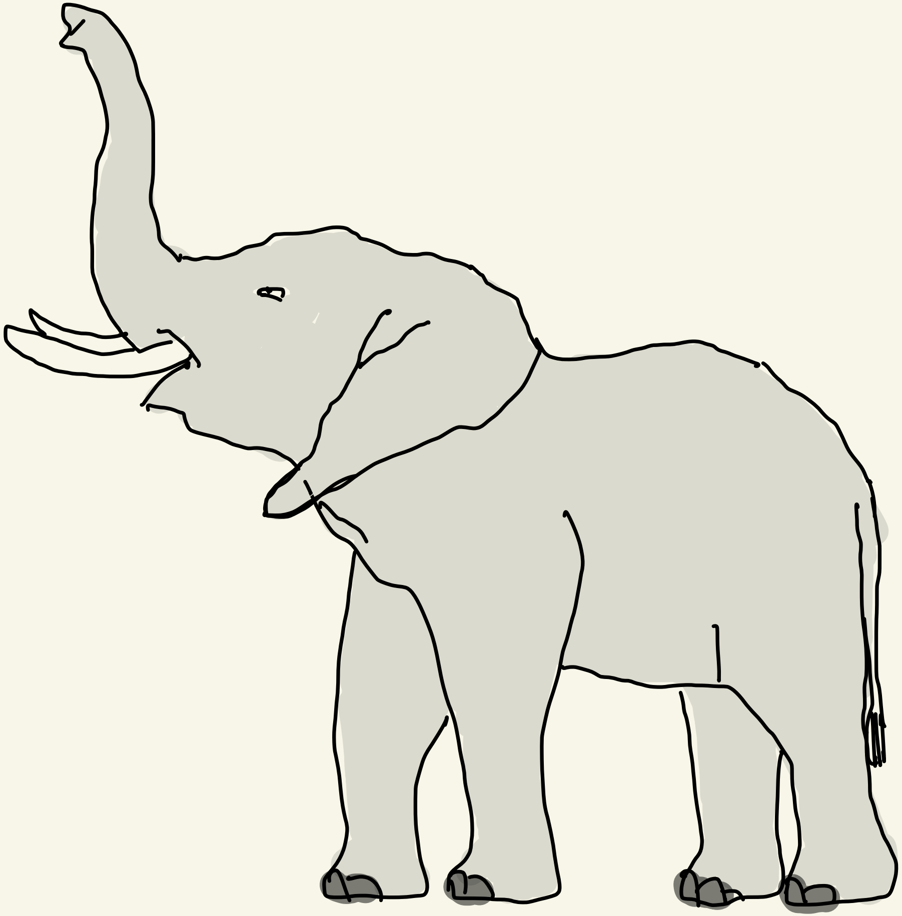
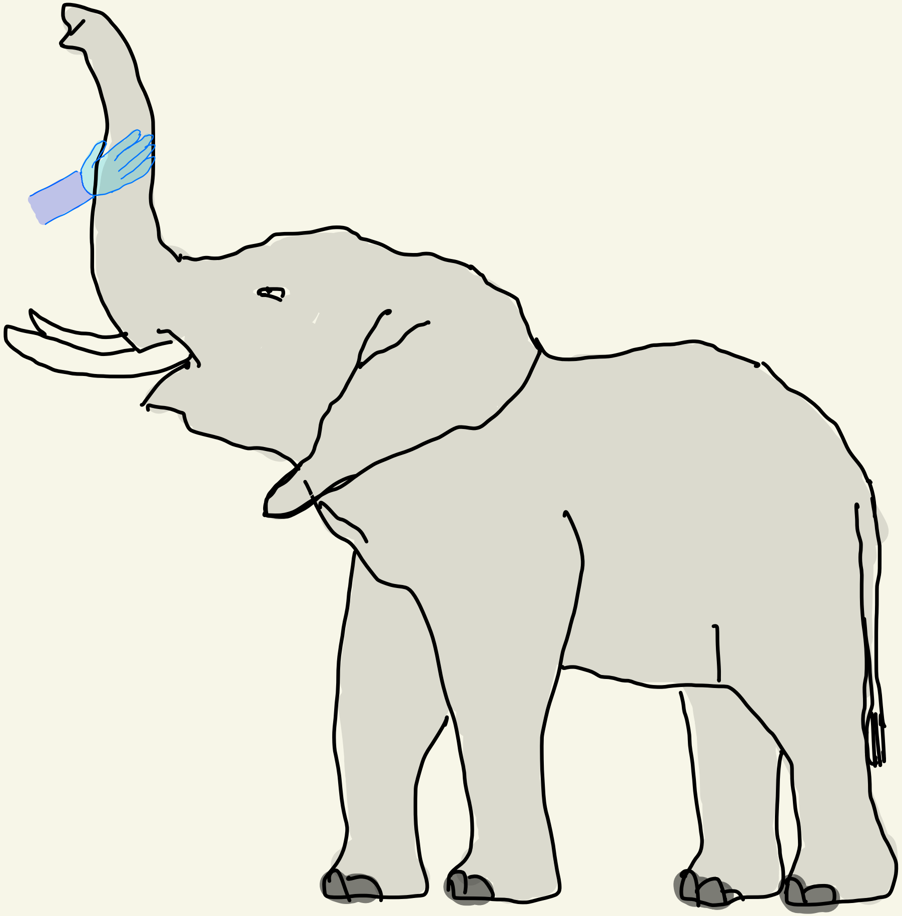
Lower bounds on the complexity of quantum proofs

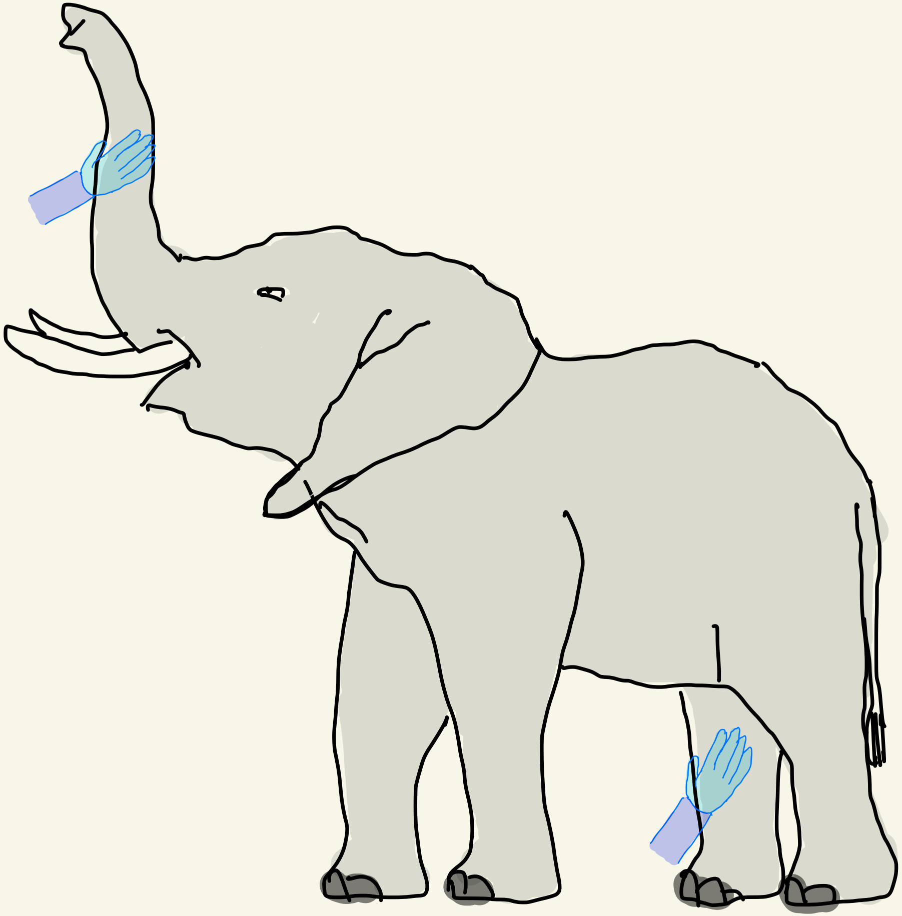
Chinmay Nirkhe

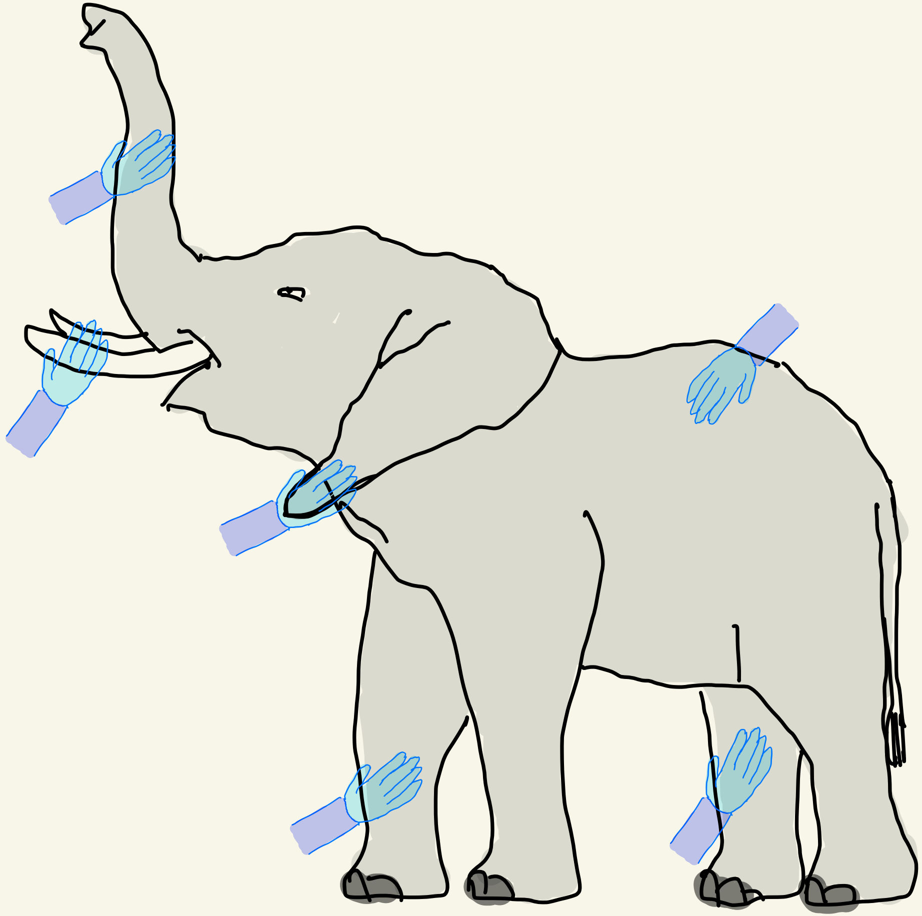
UC Berkeley

August 29th, 2022

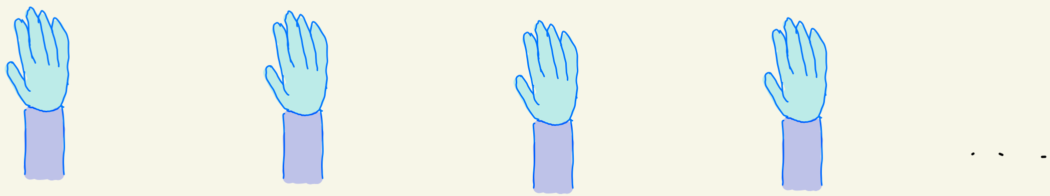


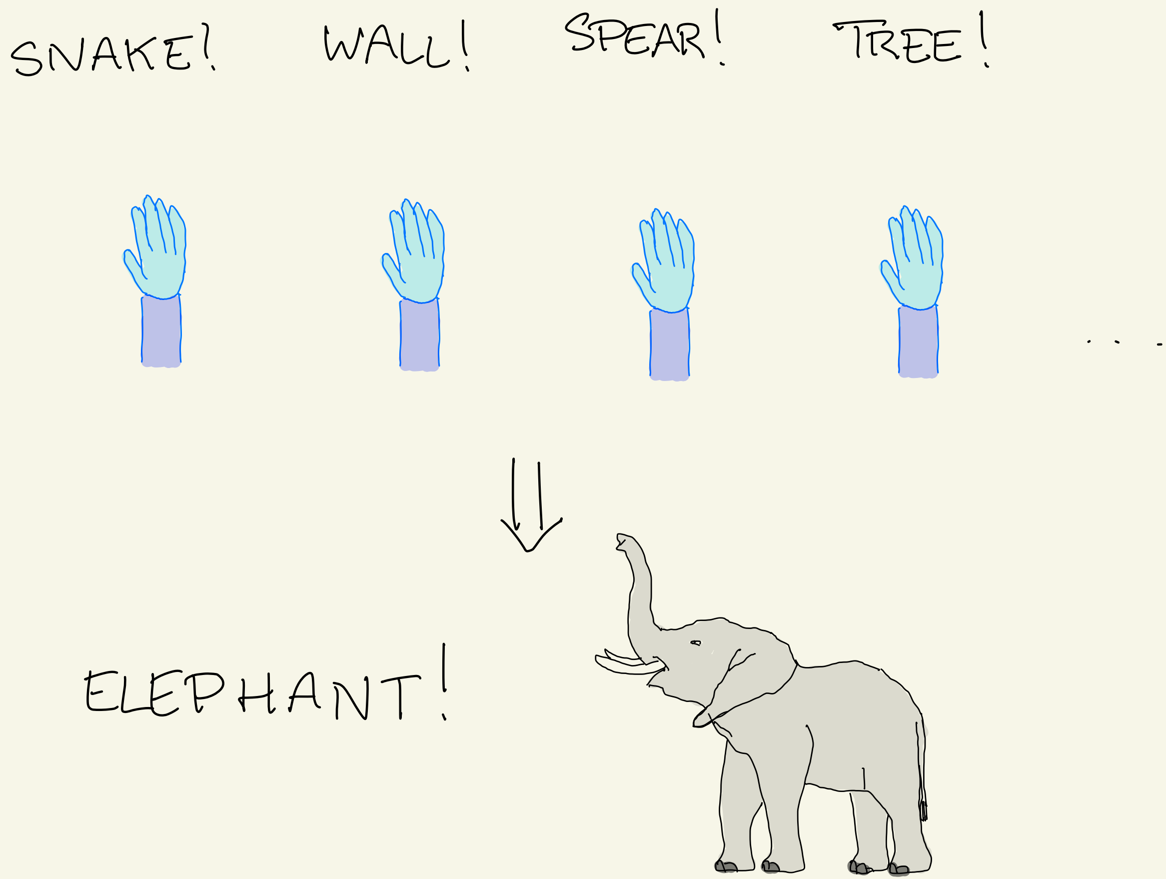






SNAKE! WALL! SPEAR! TREE!

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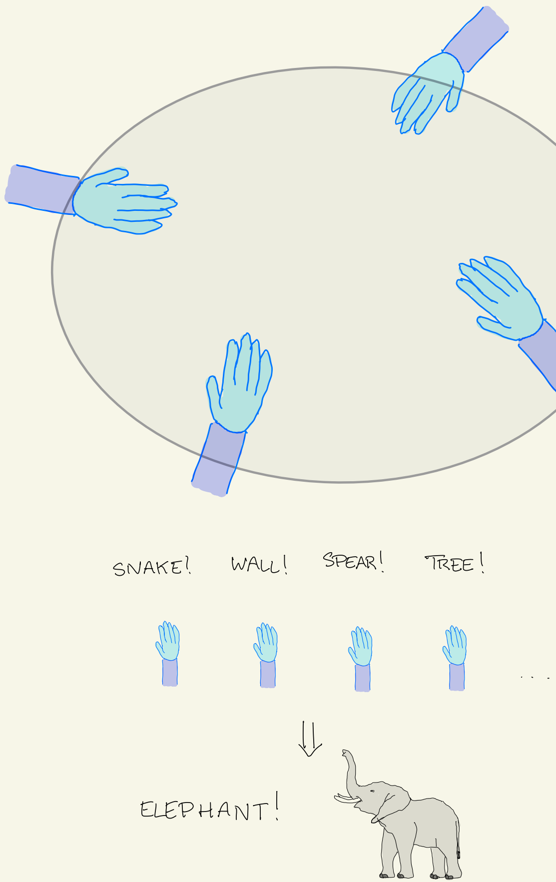
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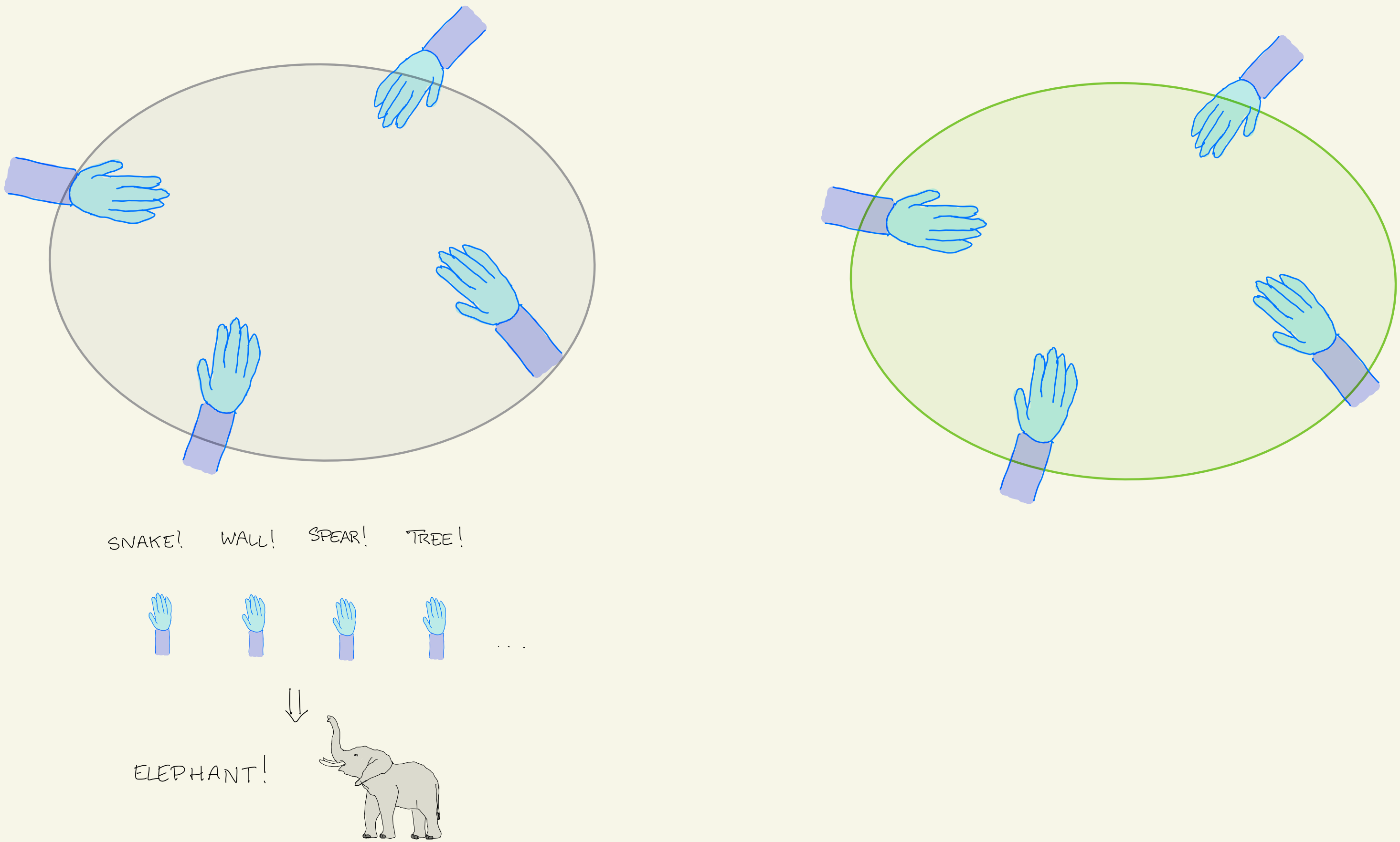
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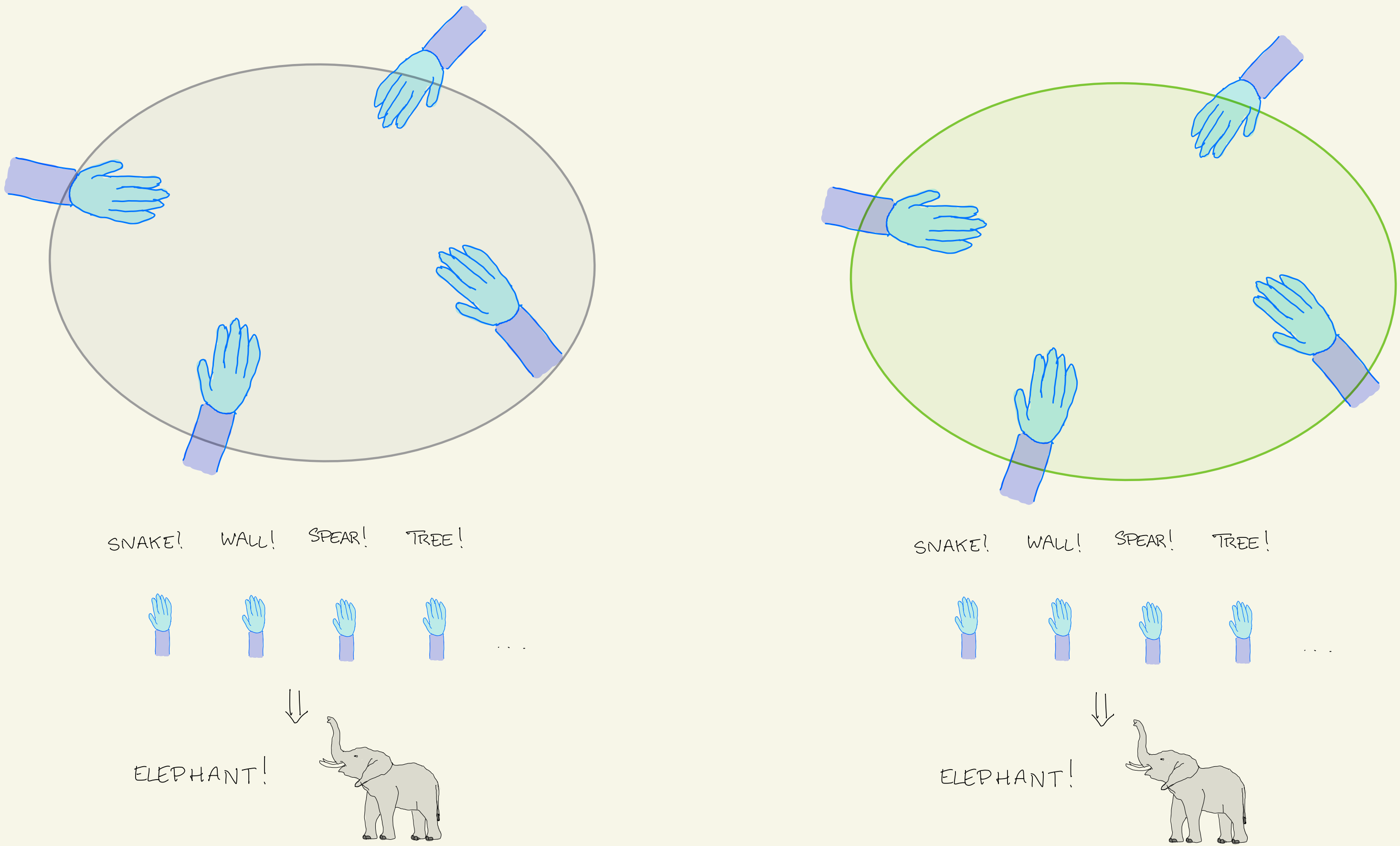
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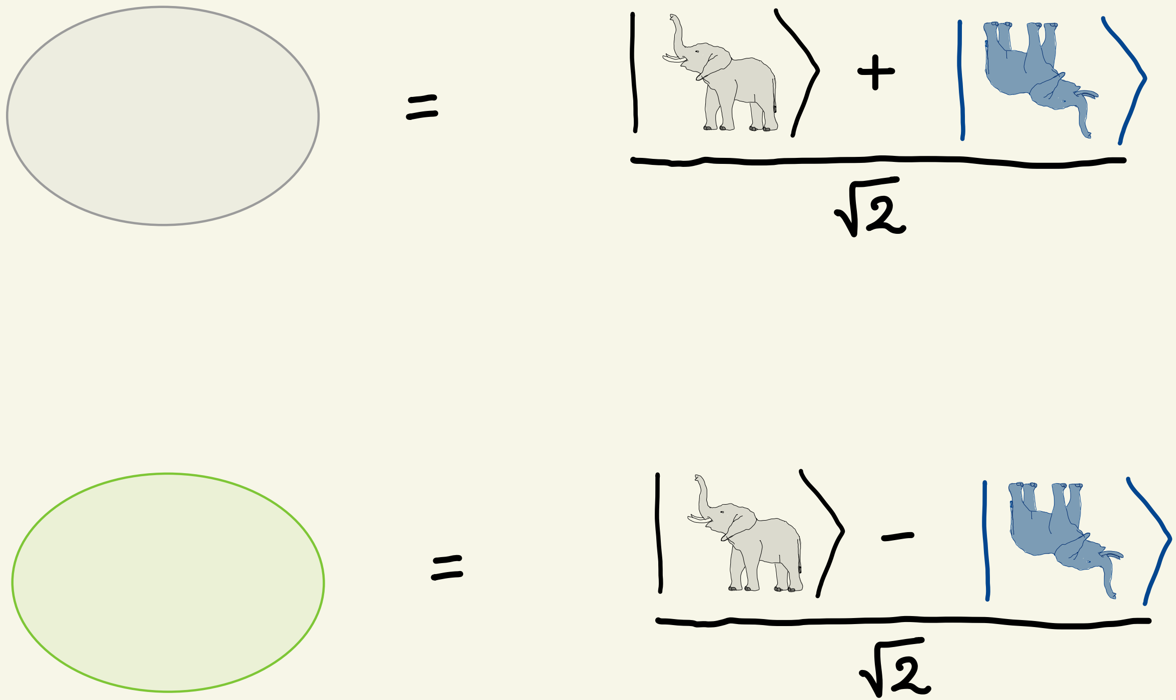
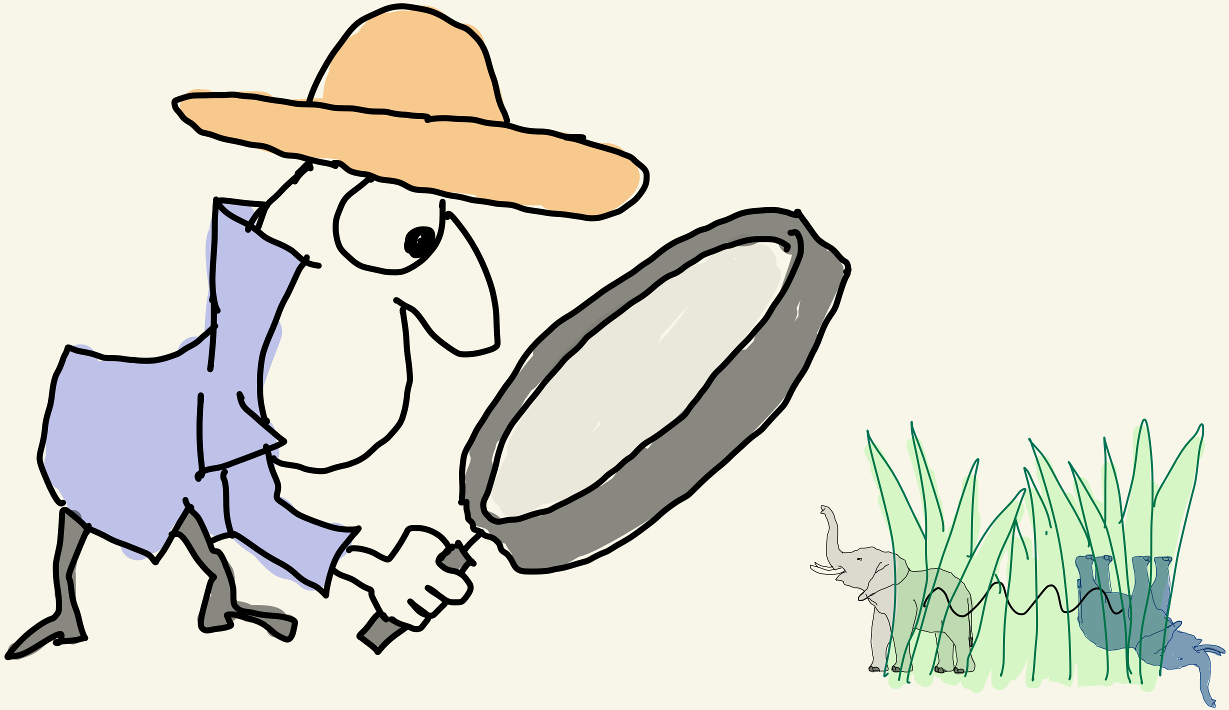
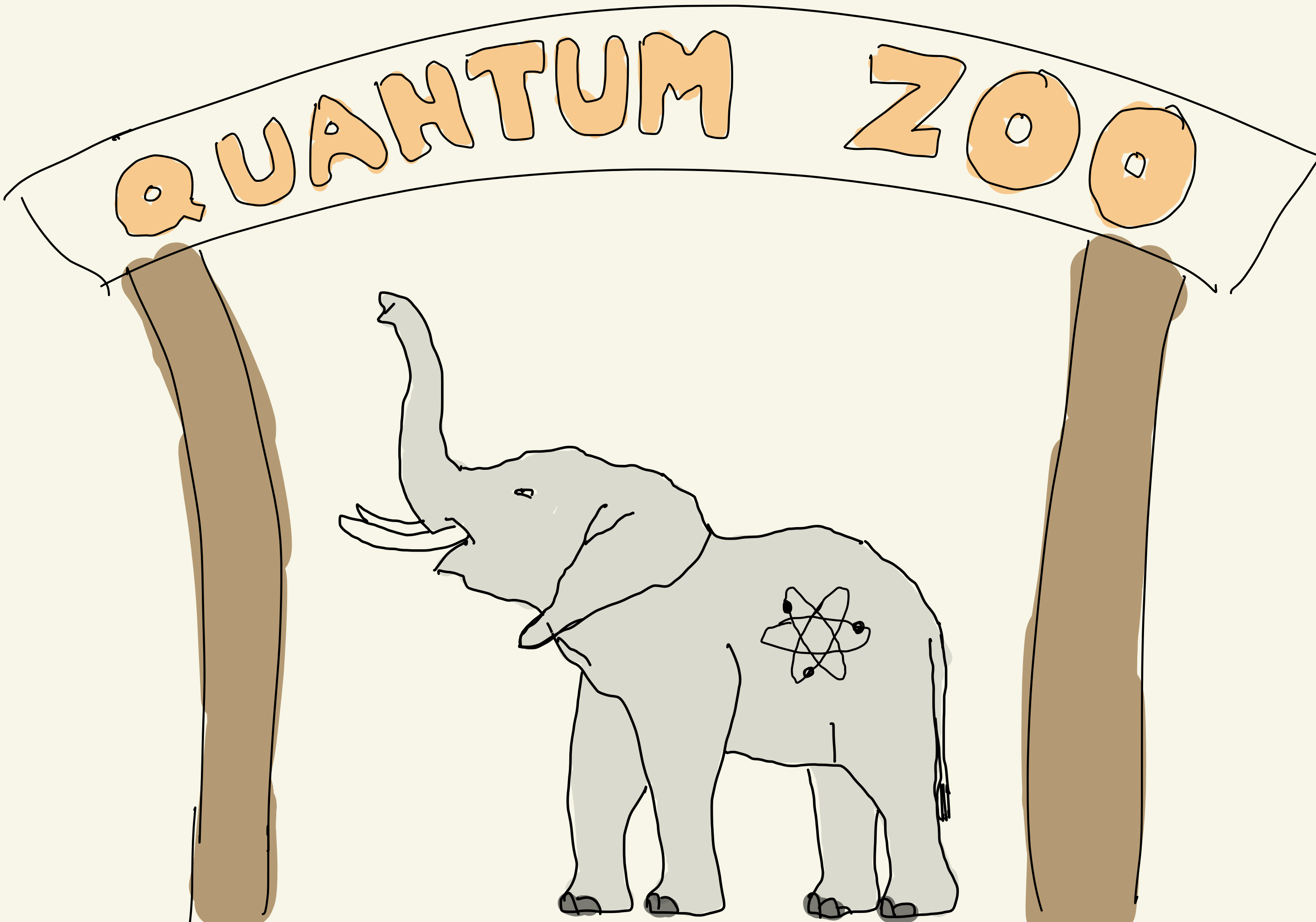
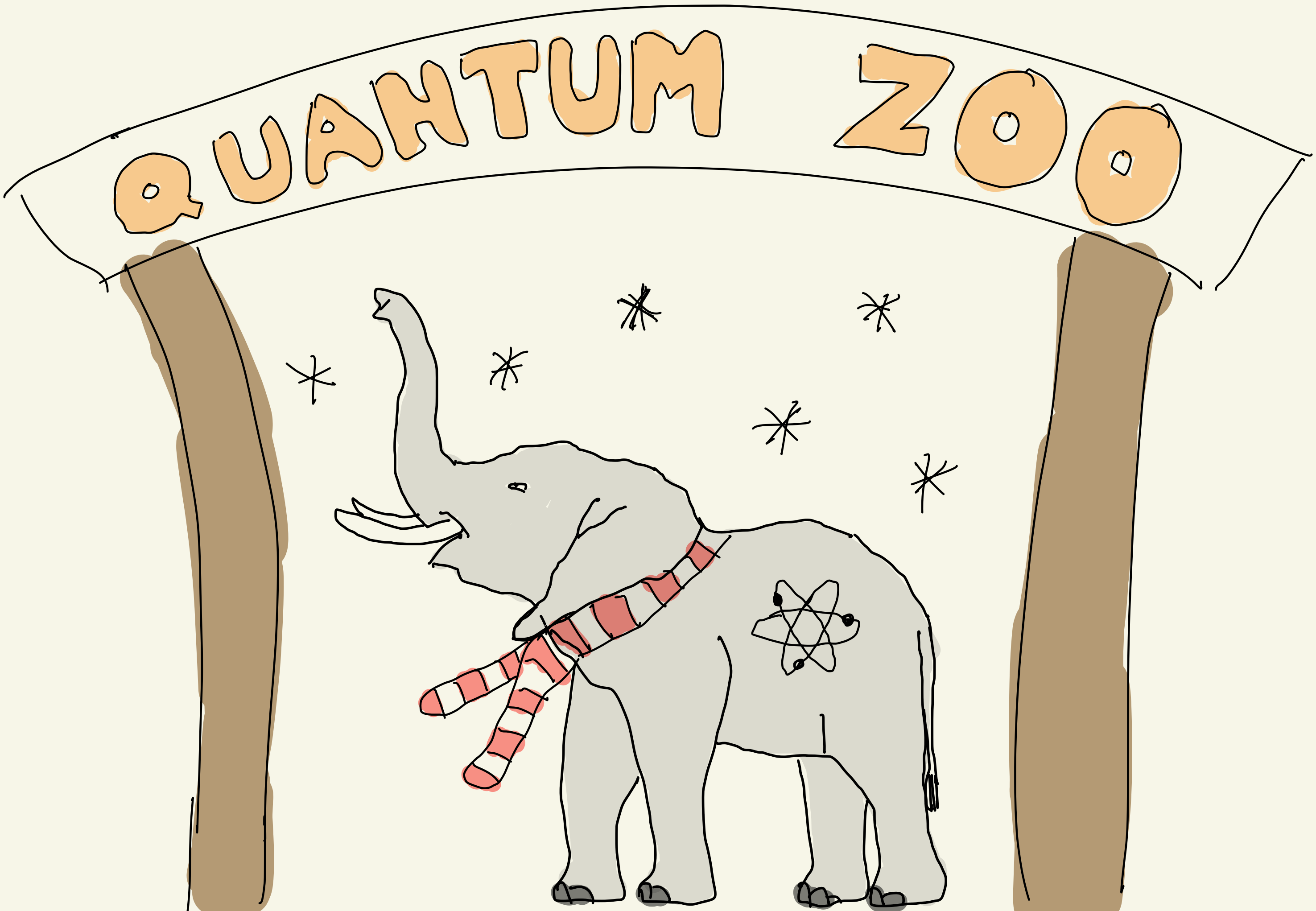





QUANTUM ZOO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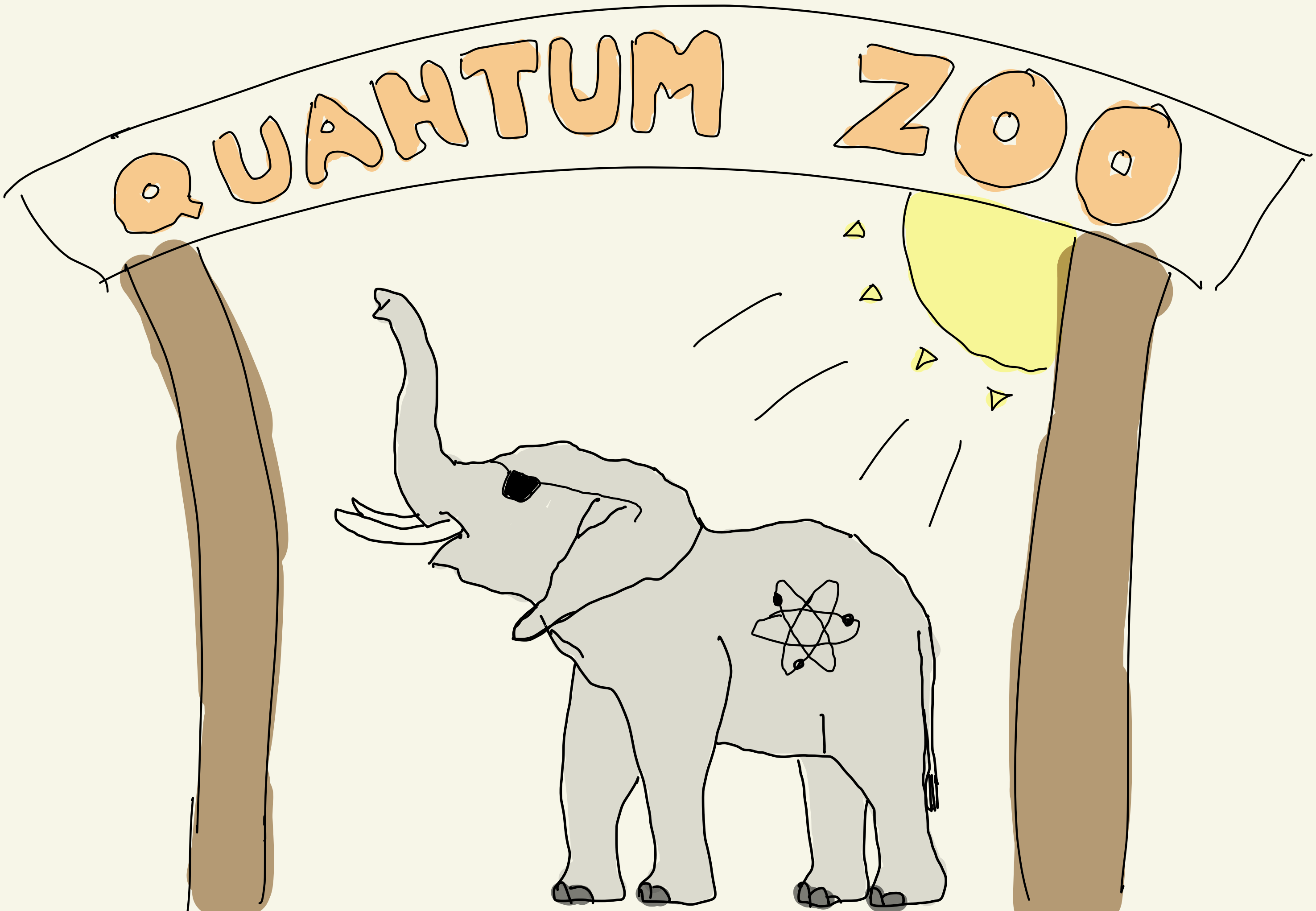
QUANTUM ZOO

And now for the actual dissertation...

Lower bounds on the complexity of quantum proofs

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Understanding classical proofs

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NP = the class of all efficiently (poly(n) time) checkable proofs.

NP has complete problems such as Constraint Satisfaction Problems (CSPs).

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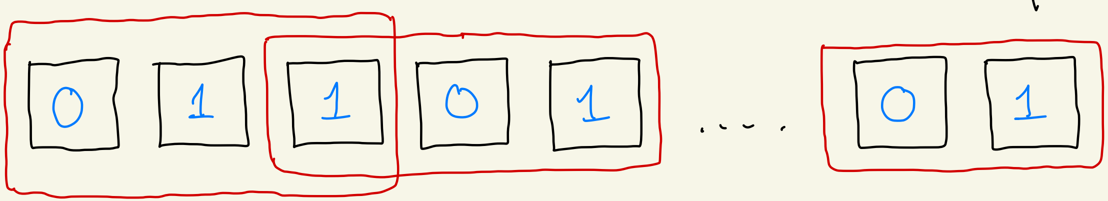

[C_i not necessarily geometrically local]

local check $C_i = x_1 \oplus x_2 \oplus x_3 = 0$.

$$C_i : \{0,1\}^3 \longrightarrow [0,1].$$

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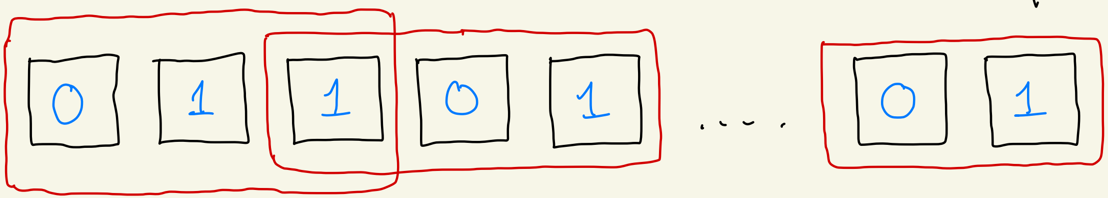

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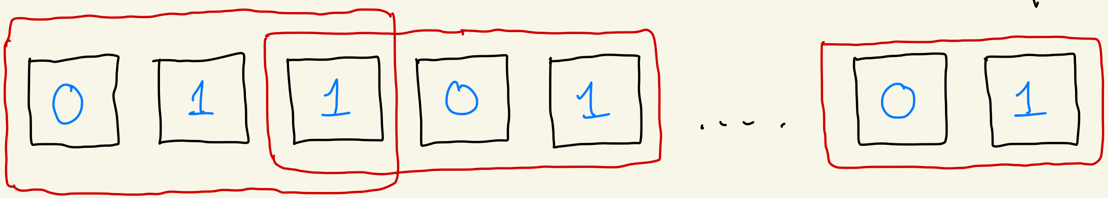

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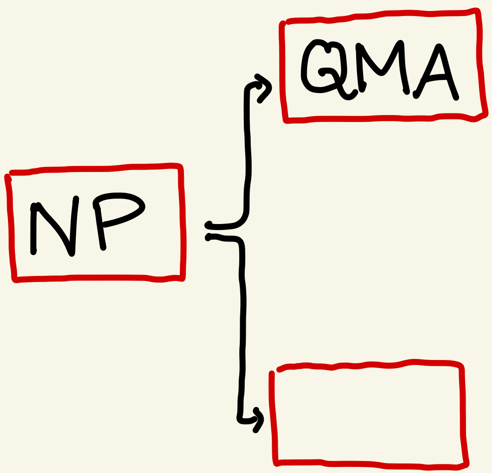
Decide if

1. $\exists\, x,\ C(x) = 0$.
2. $\forall\, x,\ C(x) \geq 1$.

Two extensions of the notion of proofs



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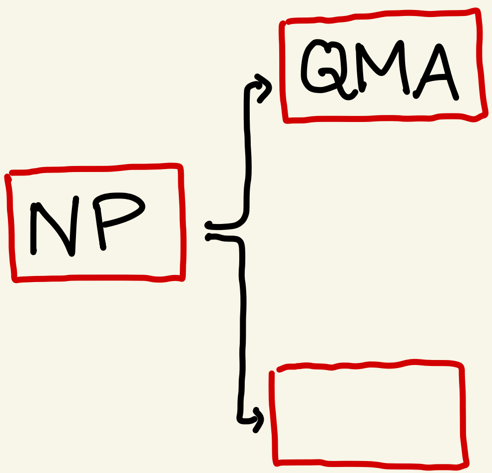

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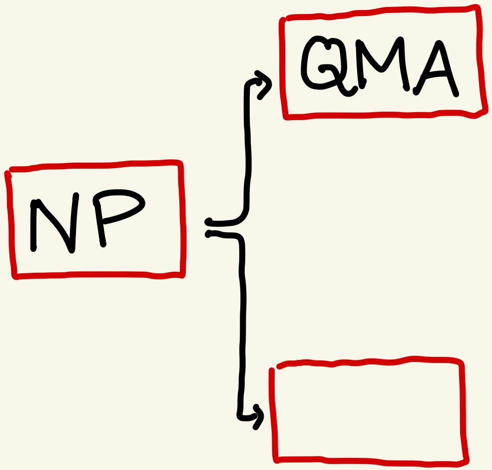

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Calculating ground energy of local Hamiltonians is a complete problem

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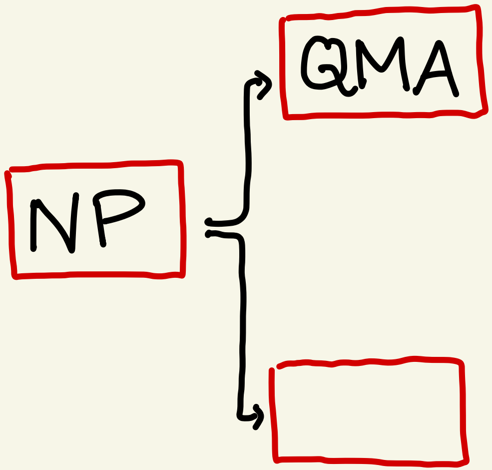

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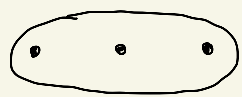
....  ... $h_i = |000\rangle\langle 000| + |111\rangle\langle 111|$

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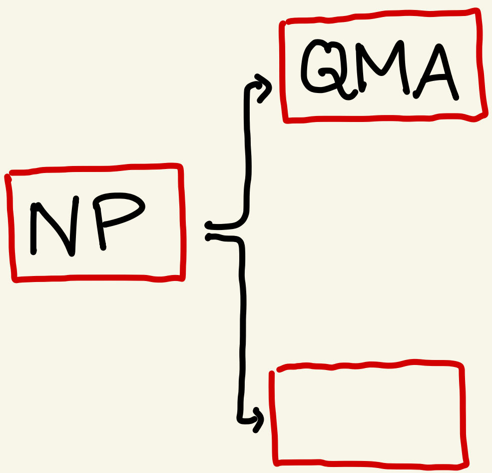

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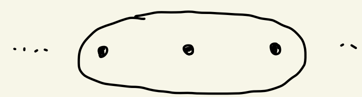

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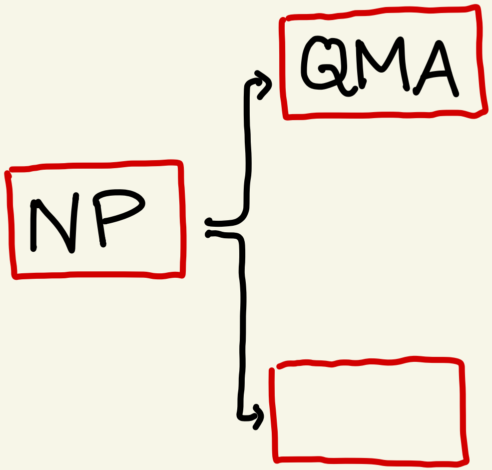

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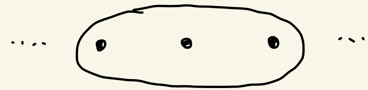

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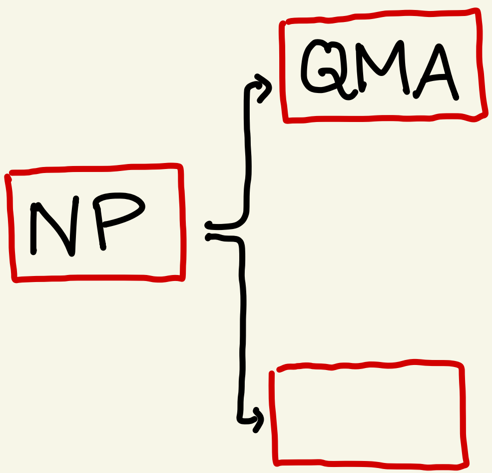

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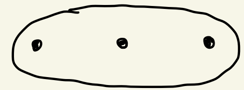
…  … $h_i = |000\rangle\langle 000| + |111\rangle\langle 111|$

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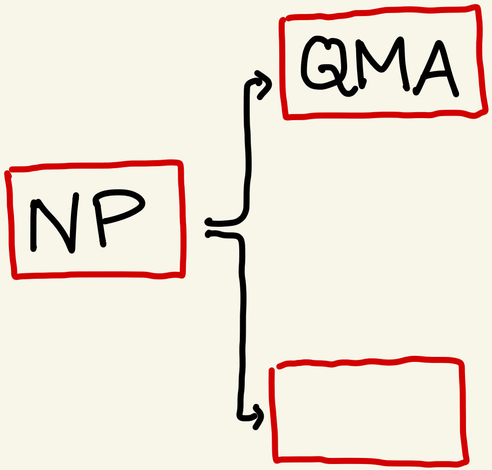

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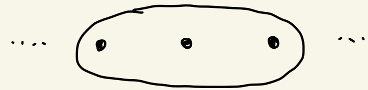

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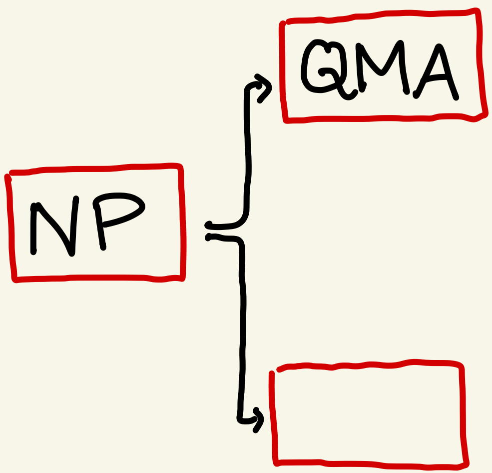

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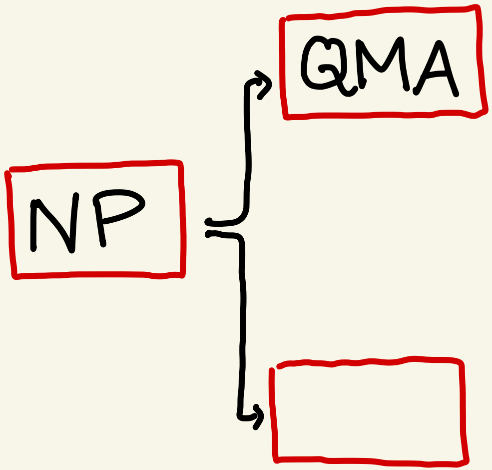

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$\implies$ groundstates of local Hamiltonians are a "canonical" form for all q. pfs.

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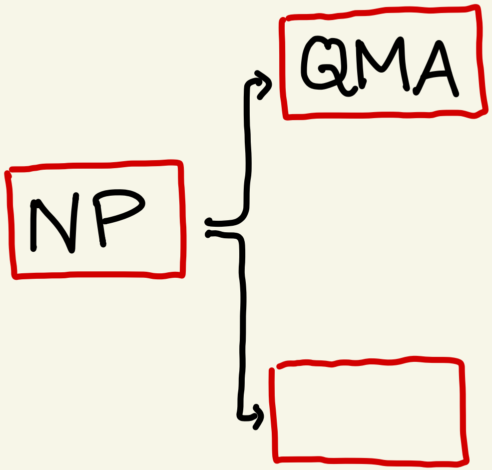

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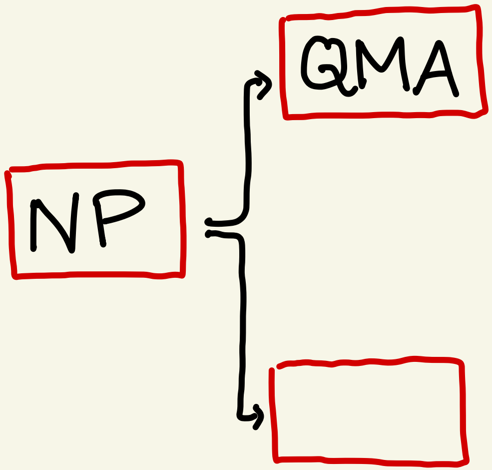

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It's widely believed that $NP \neq QMA$

$$\boxed{\text{Therefore, not all groundstates of local Hamiltonians can be \underline{classically} described (in an efficiently verifiable manner)}}$$

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$$\begin{tikzcd} & \boxed{\text{QMA}} \\ \boxed{\text{NP}} \arrow[ur] \arrow[dr] & \\ & \boxed{\text{PCPs}} \end{tikzcd}$$

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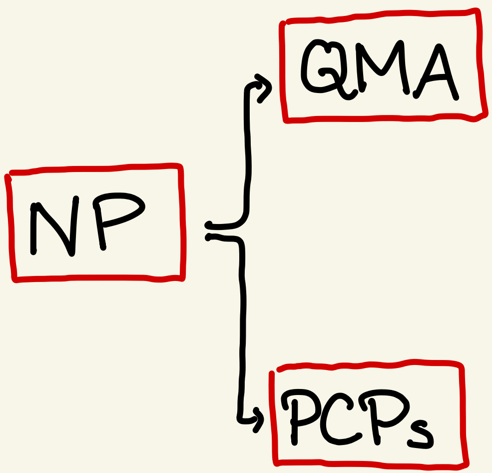
we think of pfs as requiring step-by-step checking.



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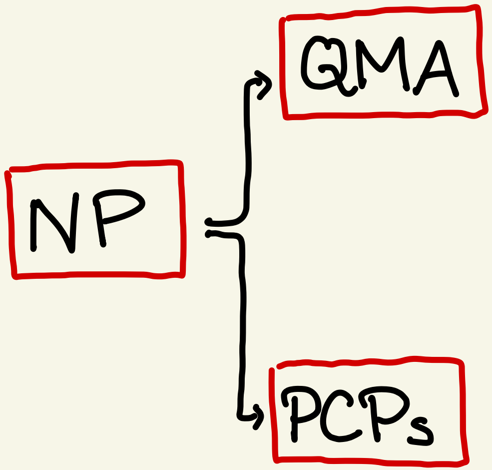

PCP theorem Every NP problem (i.e. every pf.) can be converted into a form s.t. only $O(1)$ bits need to be read to be 99% confident in validity.

Arora-Safra et.al 98, Dinur

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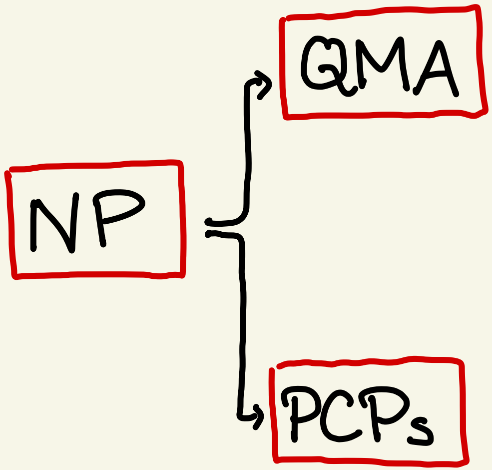

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[$C(x)$ = analog of $\langle \psi | \mathbf{H} | \psi \rangle$]

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Important consequence: $\boxed{\text{Noisy pfs suffice!}}$

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NP → QMA

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Any x s.t. $C(x) < \frac{m}{4}$ can be prob. verified with $O(1)$ queries.

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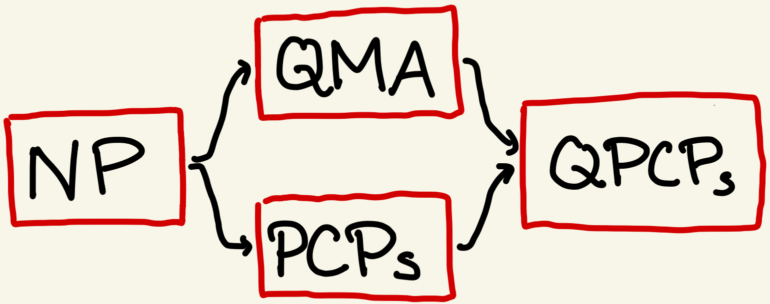
The Quantum Prob. Checkable Pfs. Conjecture



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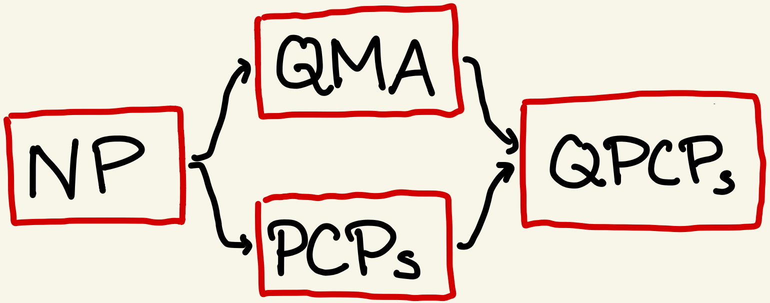

Conjecture: Every QMA problem (i.e. quantum pf!) can be converted into a form s.t. only $O(1)$ qubits need to be measured

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Conj. For $\varepsilon > 0$, it's QMA-hard to decide

① $\exists\, |\psi\rangle$ s.t. $\langle\psi|\mathbf{H}|\psi\rangle = 0$ (morally)

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Similar to PCP theorem, every state of energy $\leq \frac{\varepsilon}{2} m$ is a valid pf. for a QPCP local Hamiltonians.

Set of pfs is much larger!

An important consequence of QPCPs

(A) (if $NP \neq QMA$) quantum pfs. cannot be classically described (in any efficiently checkable manner)

(B) low. energy states of QPCP local Hamiltonians are also valid pfs (since they are noisy pfs.)

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Constant depth q. circuit descriptions are classically checkable pfs for output state

No low energy trivial states There exist local Hams. s.t. no low-energy state is the output of a constant depth circuit.

[Freedman-Hastings 14]

$$\boxed{\text{\underline{No low energy} \underline{trivial states}. There exist local Hams. s.t. \underline{no} low-energy state is the output of a constant depth circuit.}}$$

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[Freedman-Hastings 14]

- If it was false, then QPCP would have been trivially false.
- Makes a statement about physically realizable robust entanglement.

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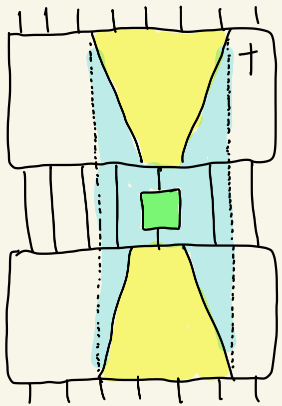
$\exists\, \varepsilon > 0$, and Hamiltonian family $\mathbf{H}$ s.t. every state ψ of energy $\le \varepsilon n$, the minimum depth circuit to generate ψ is $\Omega(\log n)$.

Proof sketch of the NLTS theorem

① Trivial states ⇒ Local Hamiltonians

⇒ Circuit depth lower bounds



Lightcones for low depth circuits

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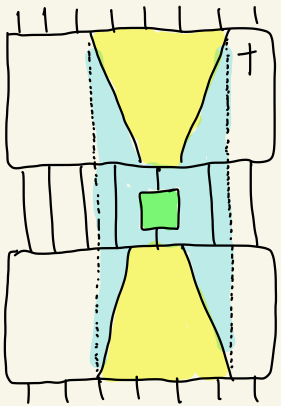
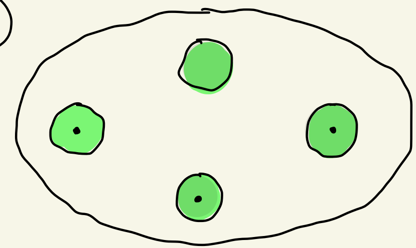


Lightcones for low depth circuits

Error Correction Codes (ECC)

②



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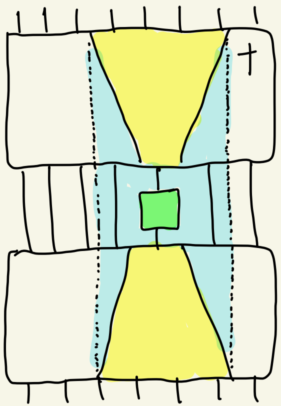

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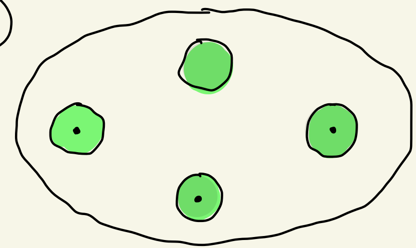

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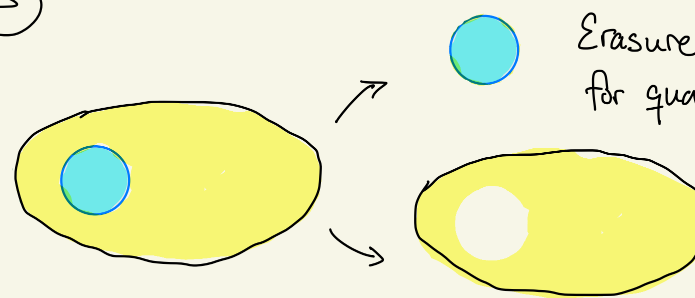

Erasure errors for quantum codes

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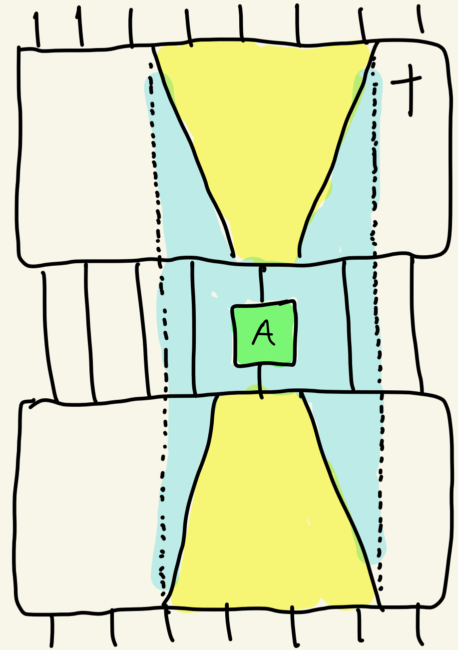

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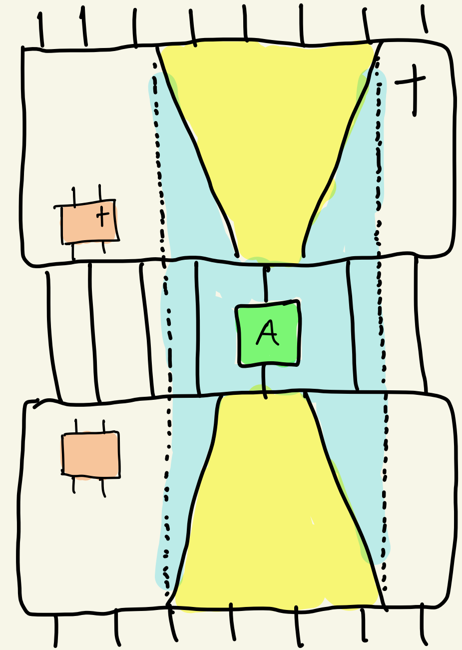

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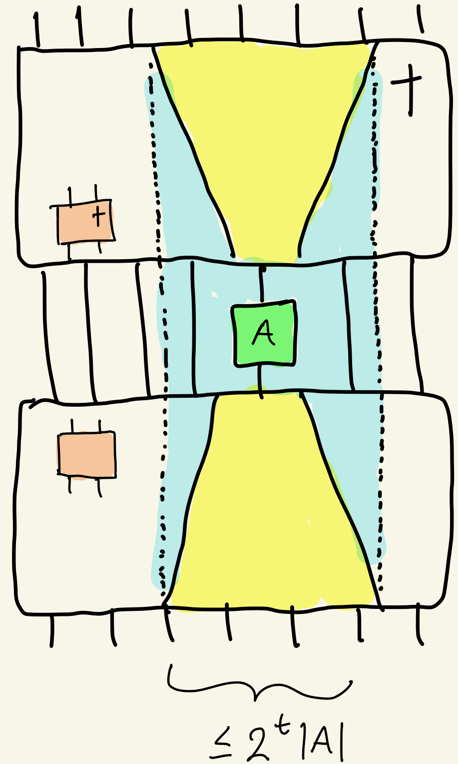

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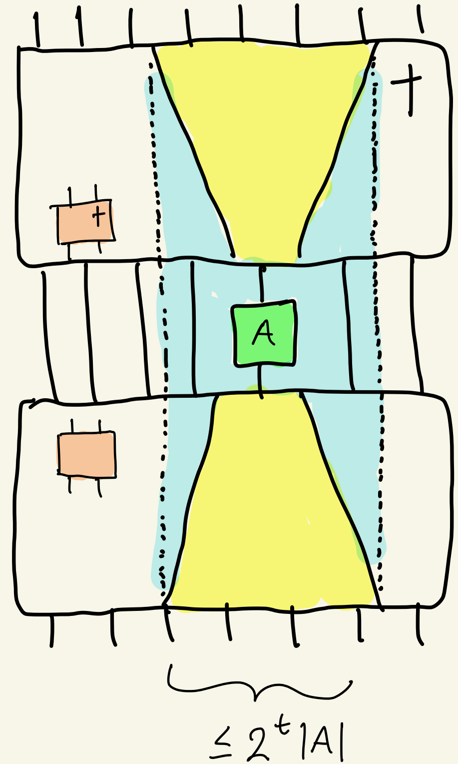

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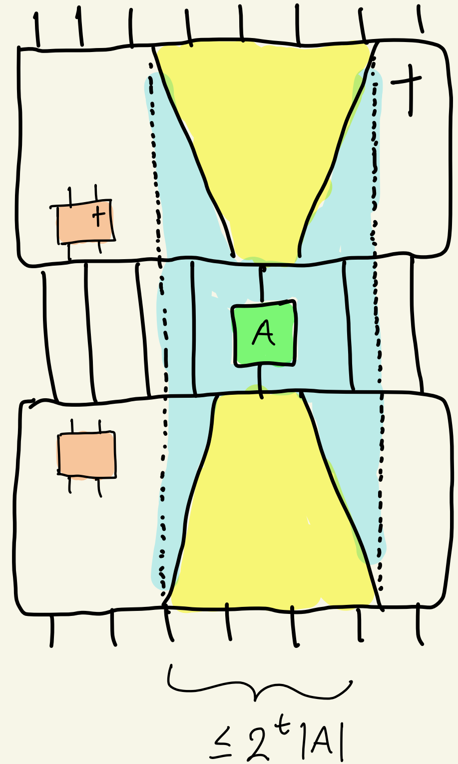

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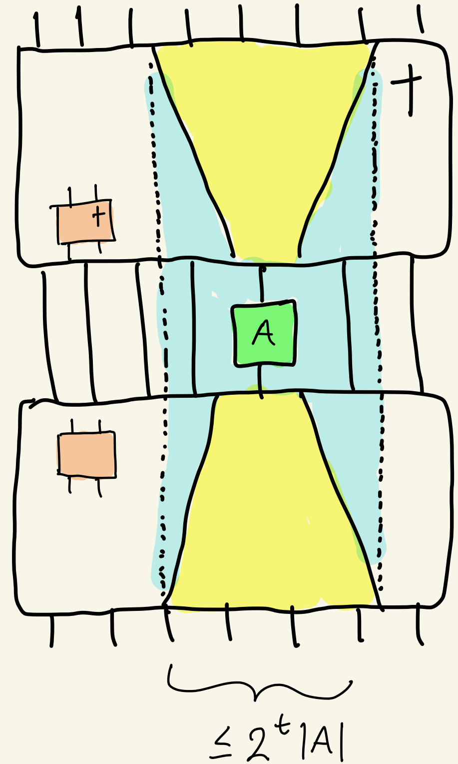

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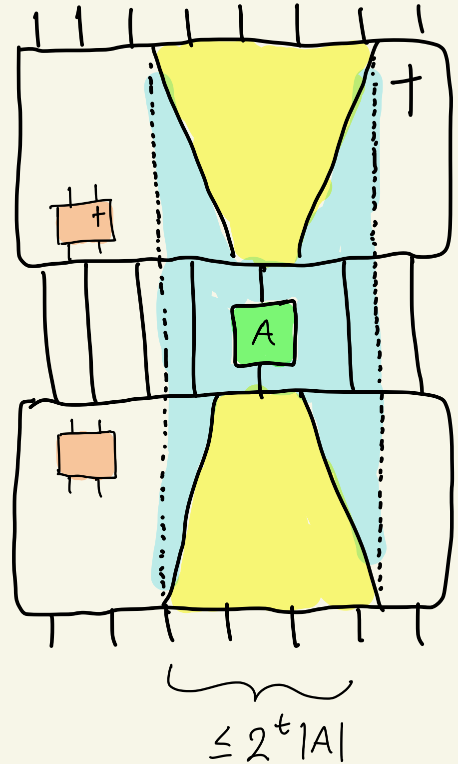

Low-depth states are classical witnesses for energy

Trivial states ⇒ Local Hamiltonians

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$\mathbf{H}_{\mathcal{U}}$ is commuting and has a spectrum of $0, 1, 2, \dots, n'$, with eigenvectors $\mathcal{U}|x\rangle$ of eigenvalue $|x|$.

And $\mathbf{H}_{\mathcal{U}}$ is a 2^t-local Hamiltonian.

Local indistinguishability

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Any strict reduced density matrix equals

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But groundstate $|\psi\rangle$ is unique! $\Rightarrow$ $|\psi\rangle = |\psi'\rangle$, a contradiction!

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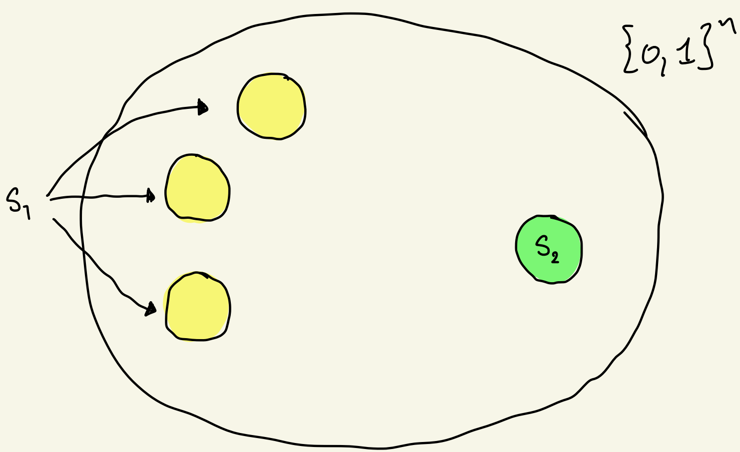

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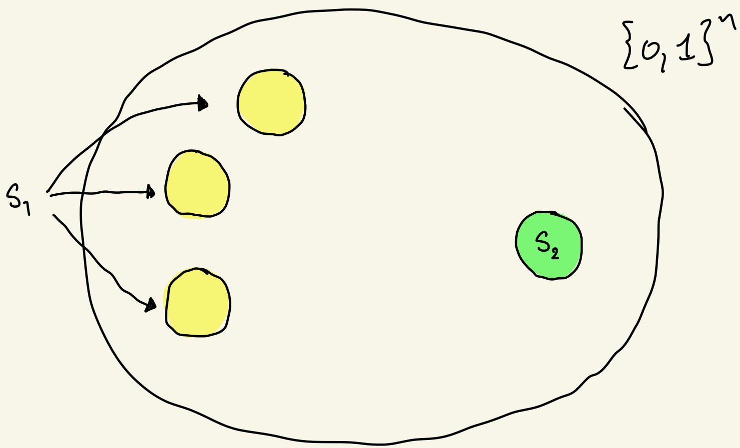

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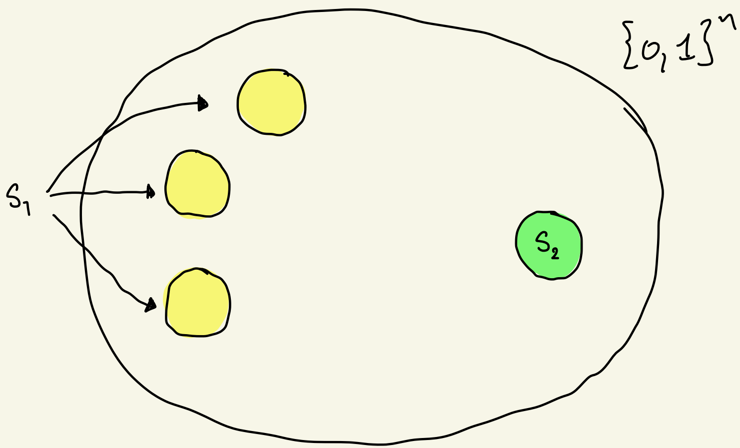

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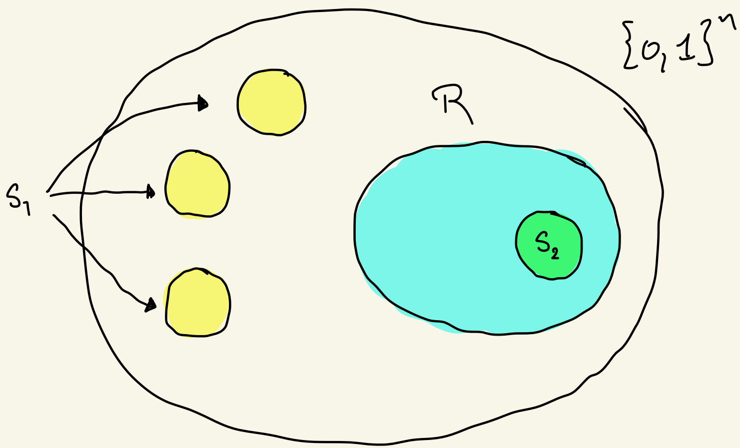

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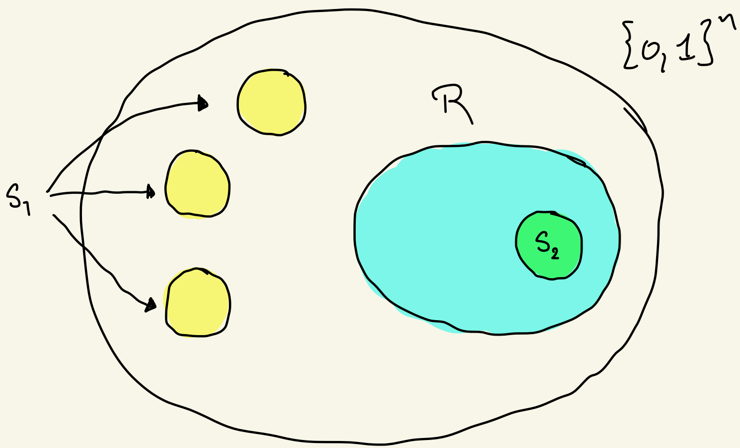

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$|\psi'\rangle$ = "flip sign of $|\psi\rangle$ on R"

and $|\psi\rangle$ and $|\psi'\rangle$ are approx. locally indistinguishable.

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When $\mathrm{dist}(S_1,S_2) \geq \omega(\sqrt{n})$ and $\mu = \Omega(1)$,

we call such distributions well spread. To prove NLTS, we need to show $\exists$ a local Hamiltonians whose entire low-energy subspace induces well-spread distributions.

Expanding codes & Tanner codes

A linear code $\subseteq \{0,1\}^n$ can be expressed as $\ker H$ for $H \in \mathbb{F}_2^{m \times n}$

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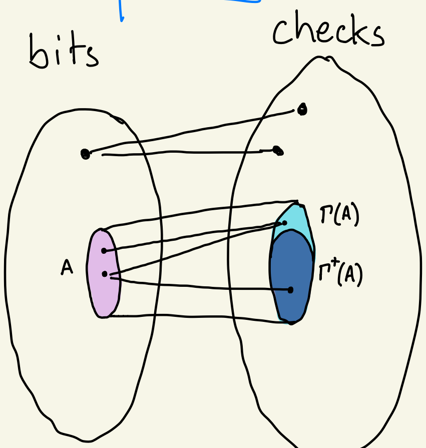

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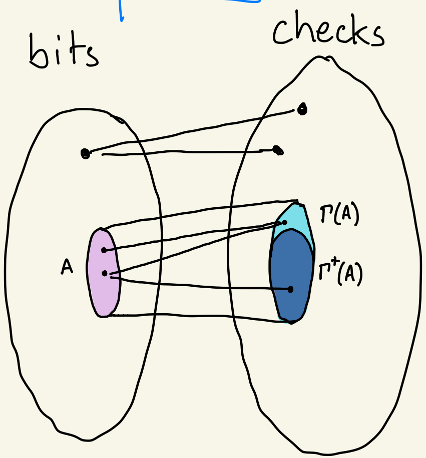

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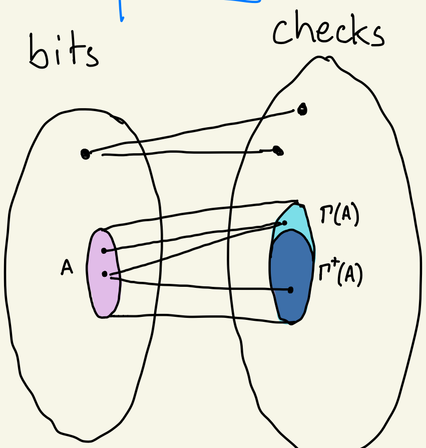

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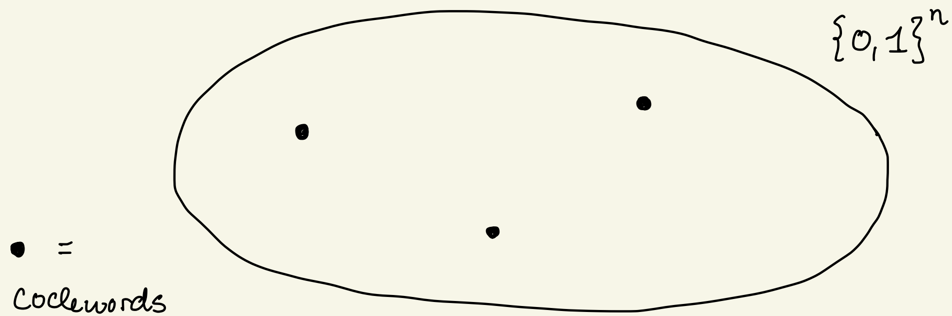

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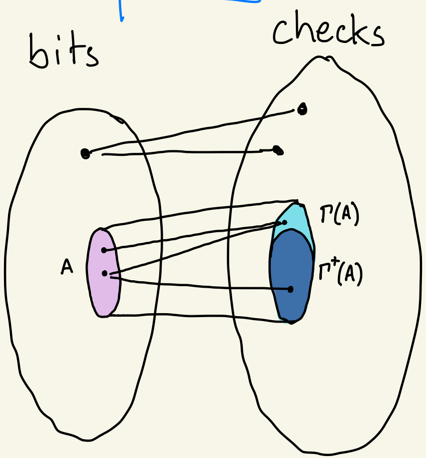

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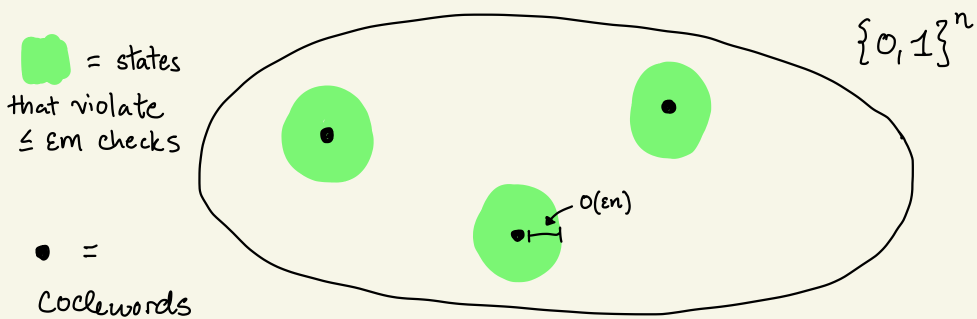

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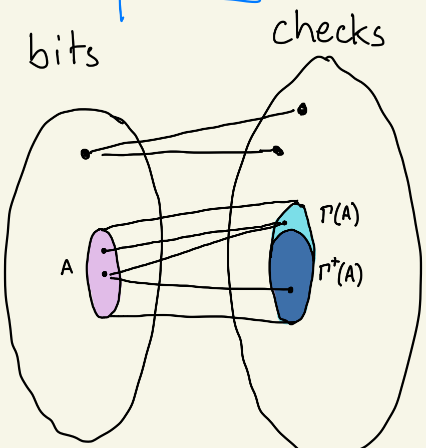

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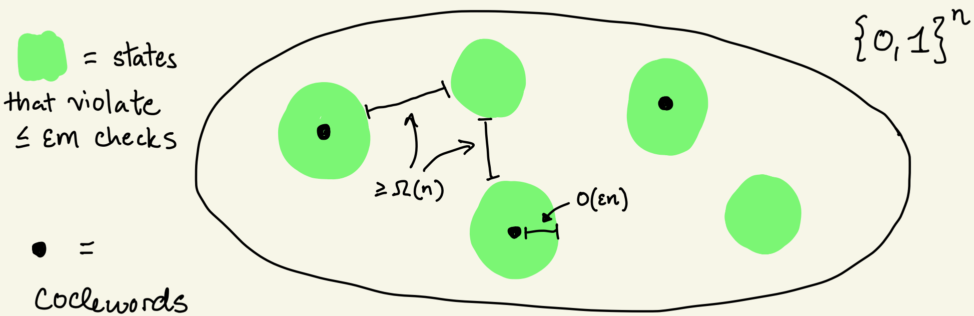

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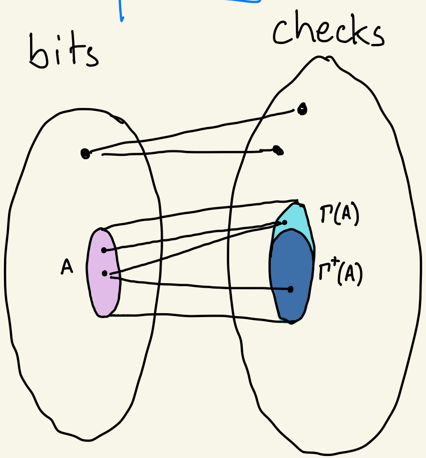

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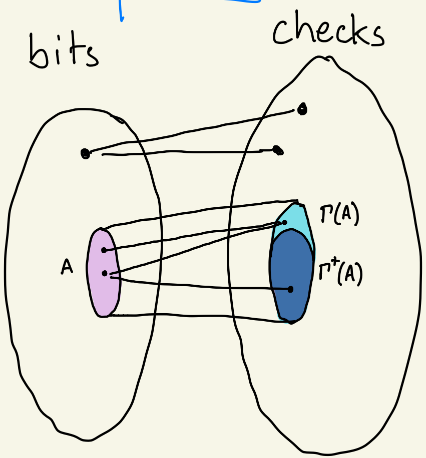

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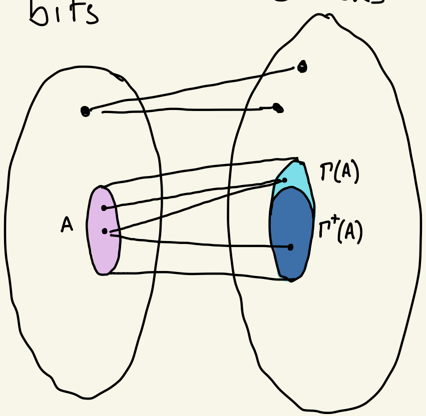

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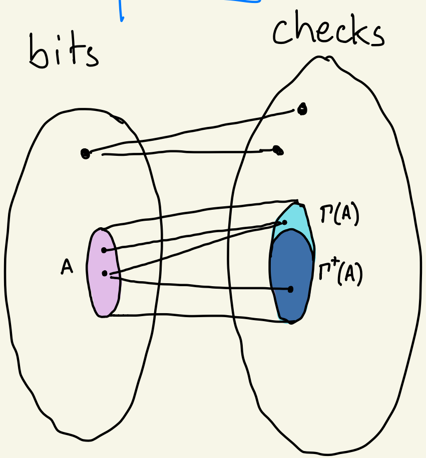

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Pf sketch: $A = \mathrm{supp}(y)$. $\Gamma^+(A)$ = unique neighbors of $|A|$.
$|\Gamma^+(A)| \geq (1-2\gamma) d |A|$. Every check in $\Gamma^+(A)$ will flag. So $|Hy| \geq (1-2\gamma) d |y|$ unless $|y| \geq c_2 n$.

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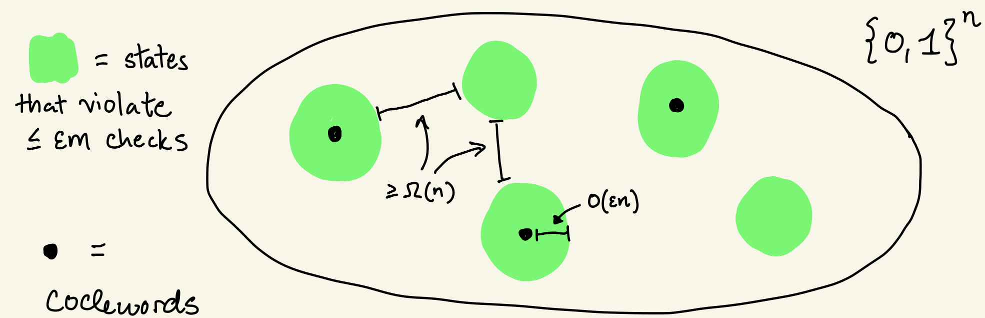

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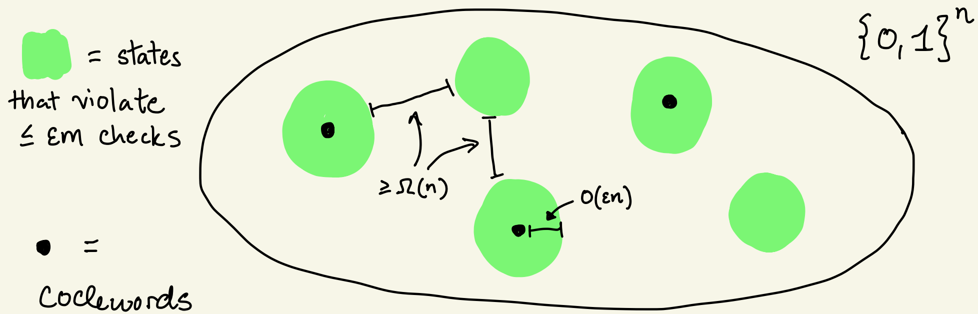

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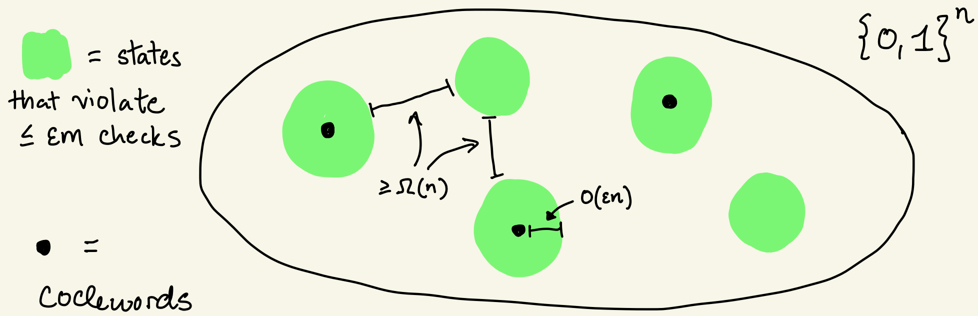
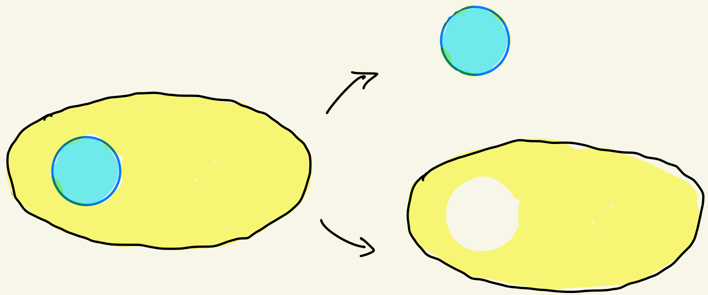


Only question is how to construct Hamiltonian with such property?

Quantum error correcting codes



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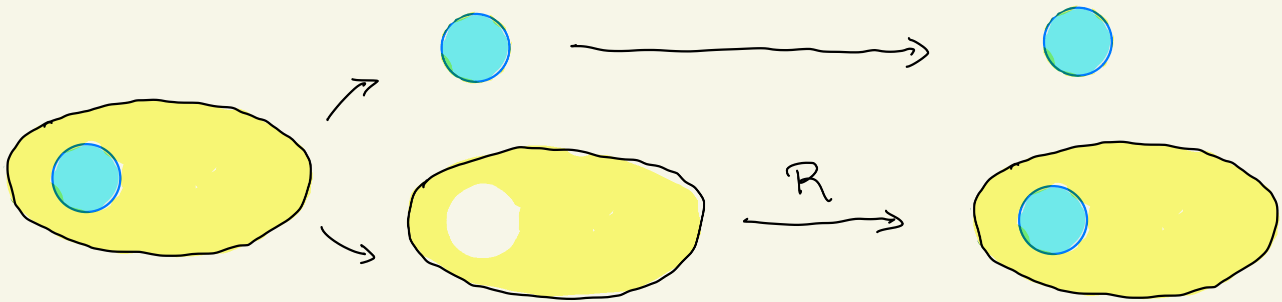

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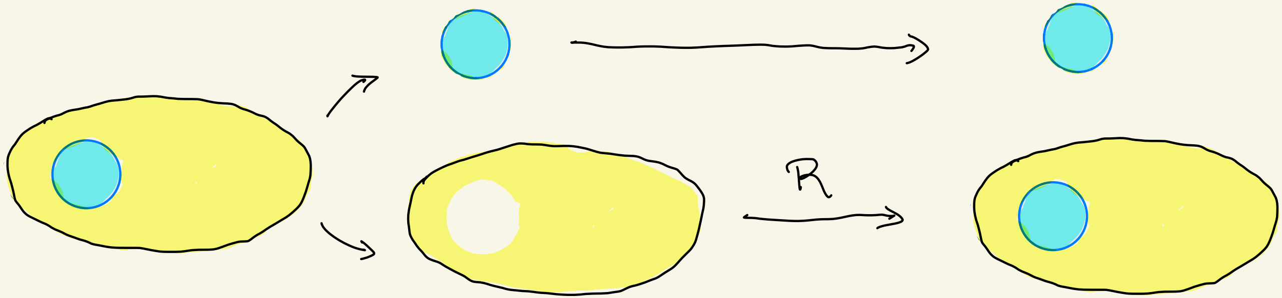

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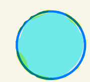
If we could recover the original state then unless  contains no information about the original state, this violates the no-cloning theorem.

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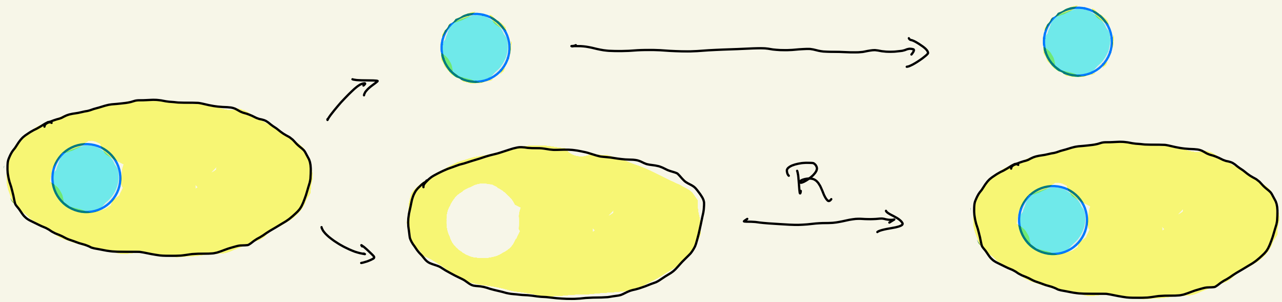

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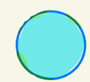
If we could recover the original state then unless  contains no information about the original state, this violates the no-cloning theorem.

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How do we prove circuit depth lower bounds for the low-energy subspace of these code Hamiltonians?

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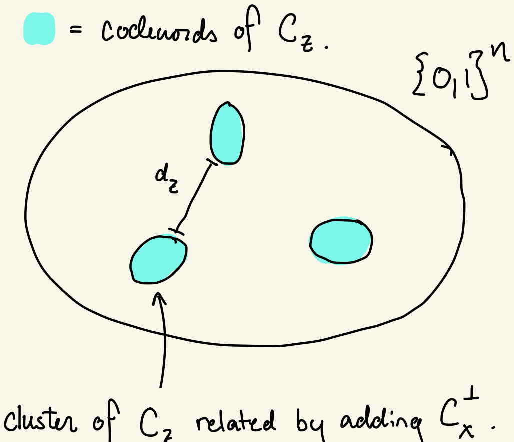

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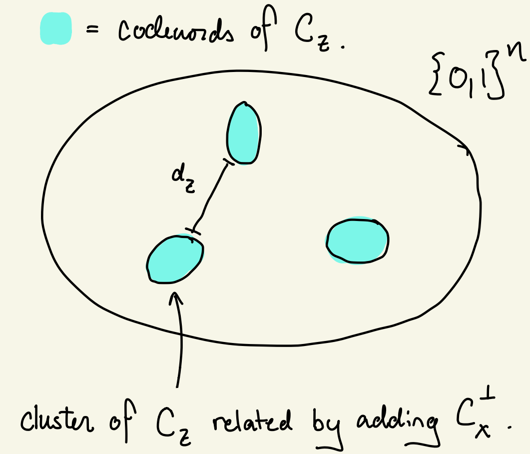

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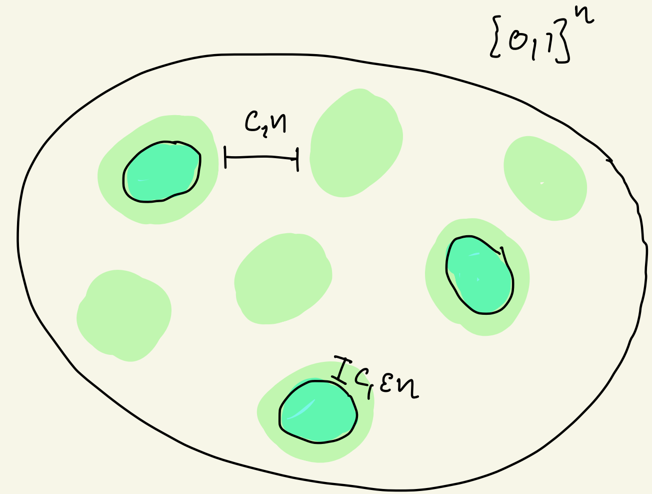

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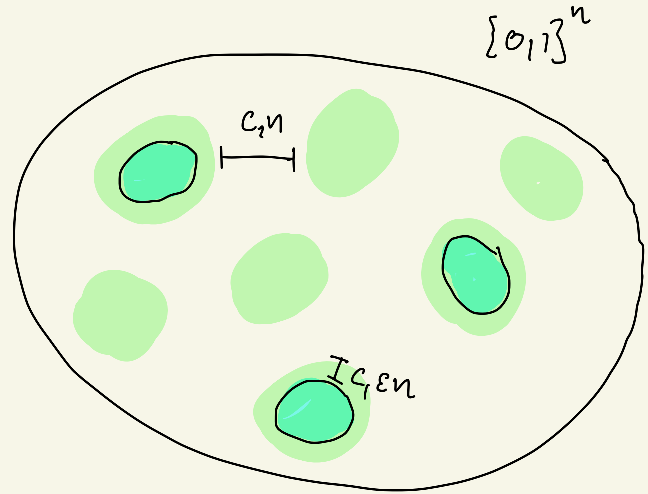
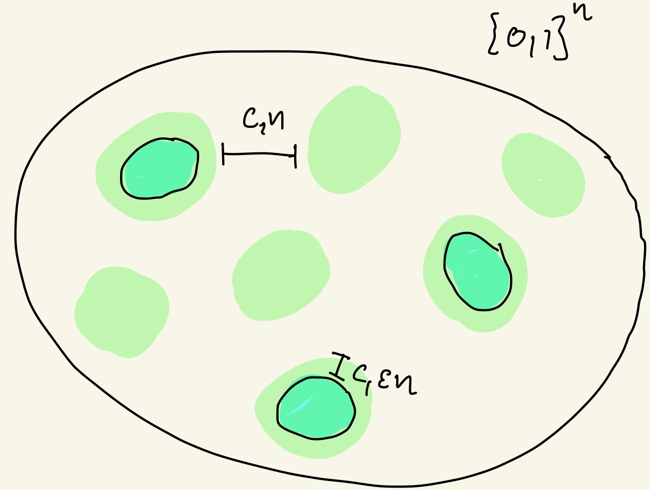


And, if we consider a $\frac{\varepsilon}{200}$-low-energy state of the code's local Hamiltonian, measuring in the Z-basis yields a dist. 99.5% supported on .

The uncertainty principle



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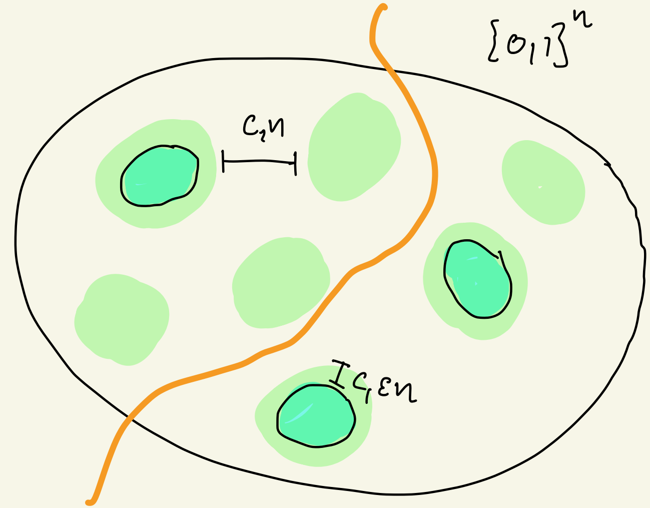

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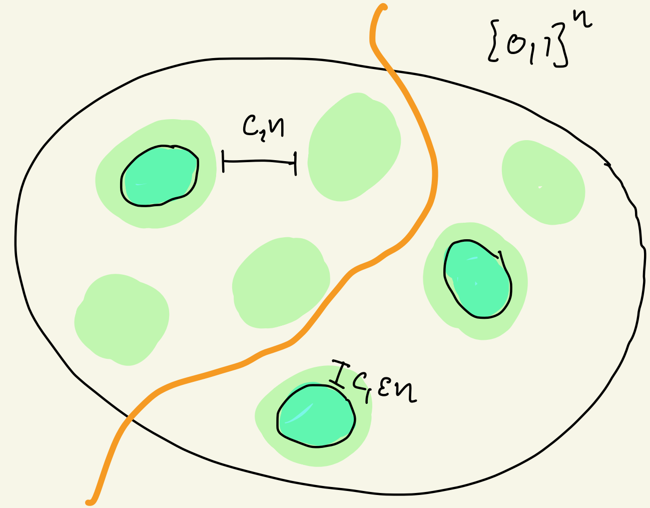

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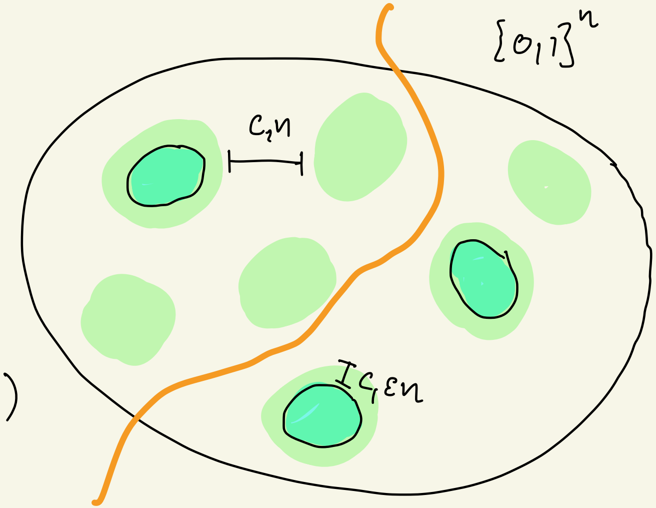

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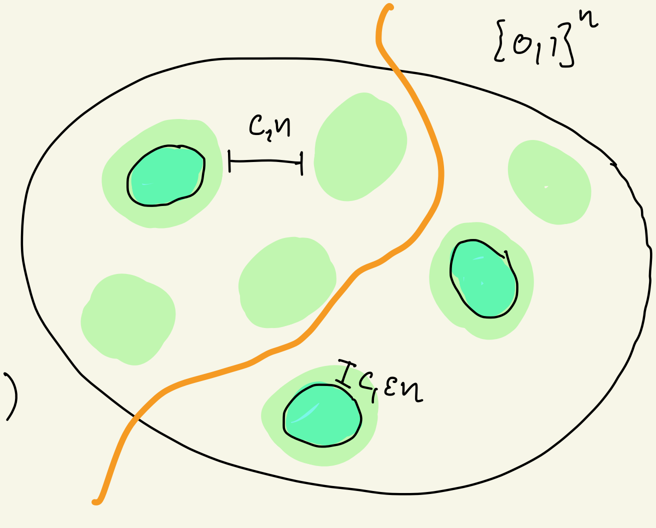

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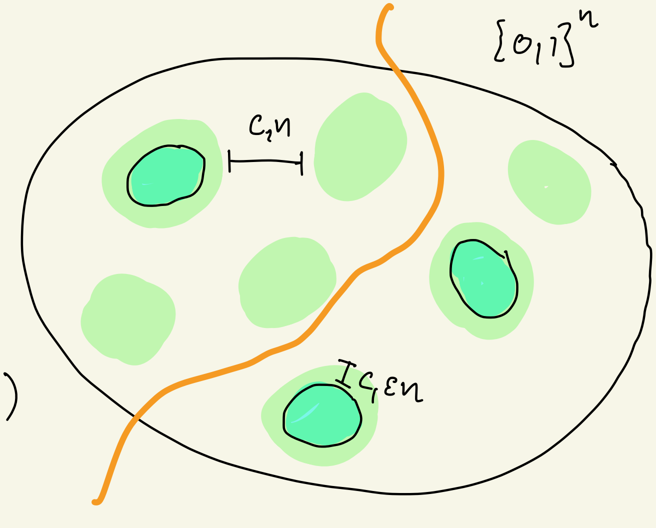

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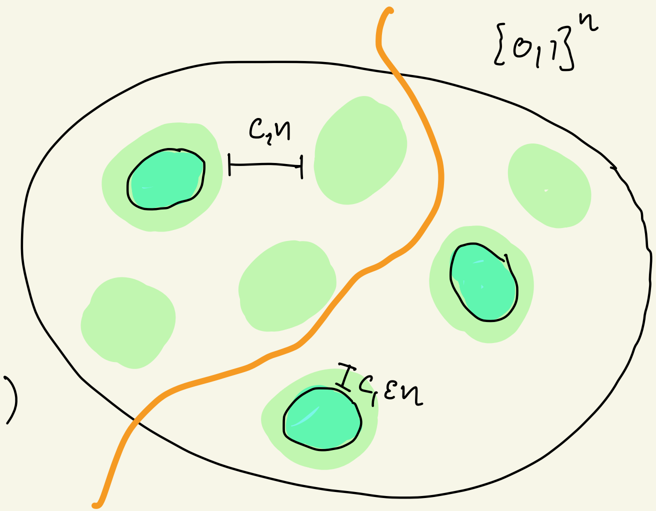

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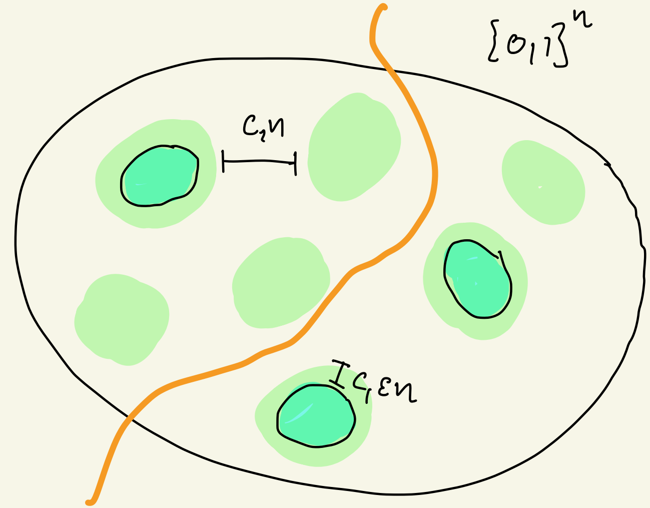

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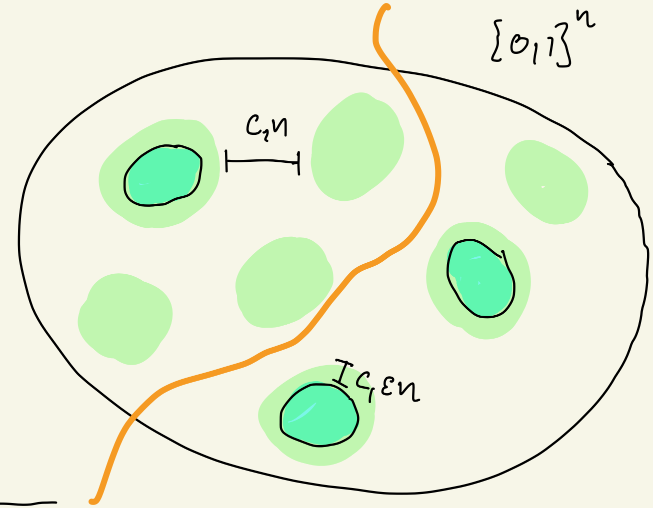

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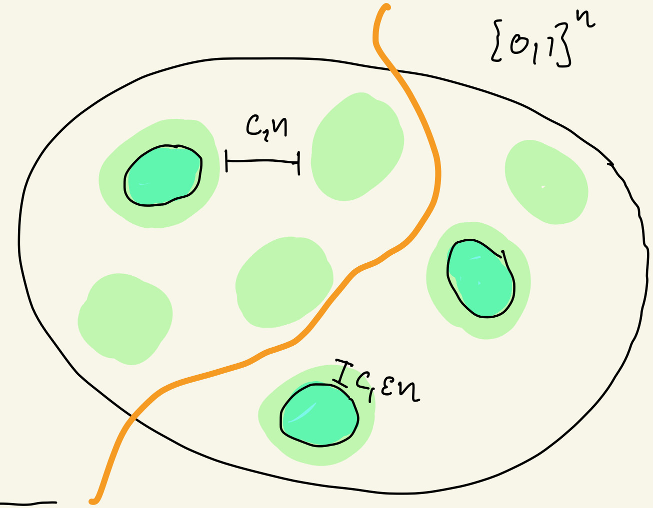

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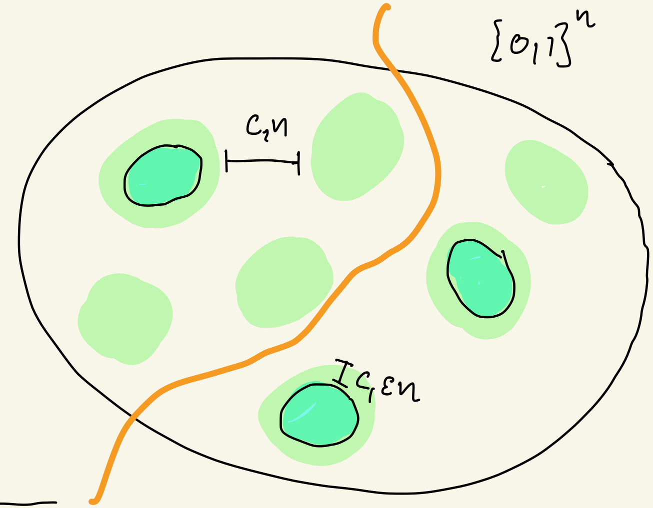

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Code rate

So if $\varepsilon < O\left(\frac{k^2}{n^2}\right)$, then $D_x(T) < 0.99$.

Uncertainty principle: For sets $S, T \subseteq \{0,1\}^n$, any state Ψ with dists. D_x, D_z

$$D_x(T) \leq 2\sqrt{1 - D_z(S)} + \sqrt{\frac{|S| \cdot |T|}{2^n}}$$

Assume D_z is $\geq 99\%$ concentrated on some Z-cluster S. Then for any X-cluster T, $D_x(T) < 0.99$ $\Rightarrow$ Either D_x or D_z is well-spread.

Conclusion of the proof

CSS code of linear-rate and linear-distance which are expanding are NLTS.

any state violating ϵn checks cannot be the output of a constant depth ckt.

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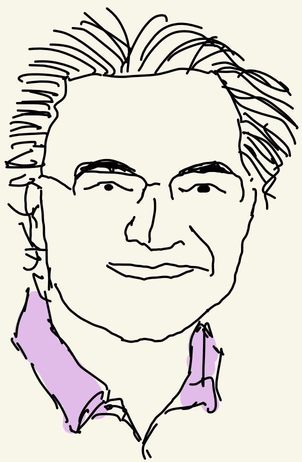
QPCP conjecture implications

1. Much harder to disprove QPCP now!

2. We need a stronger classical ansatz for classical proofs of local Hamiltonians.

Acknowledgments

Acknowledgments: Incredible Advisors



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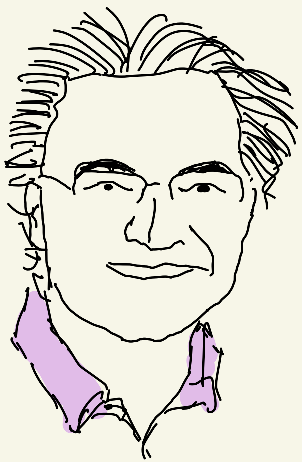

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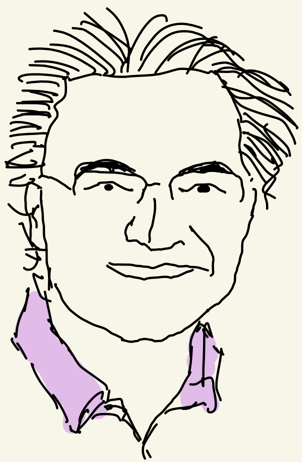

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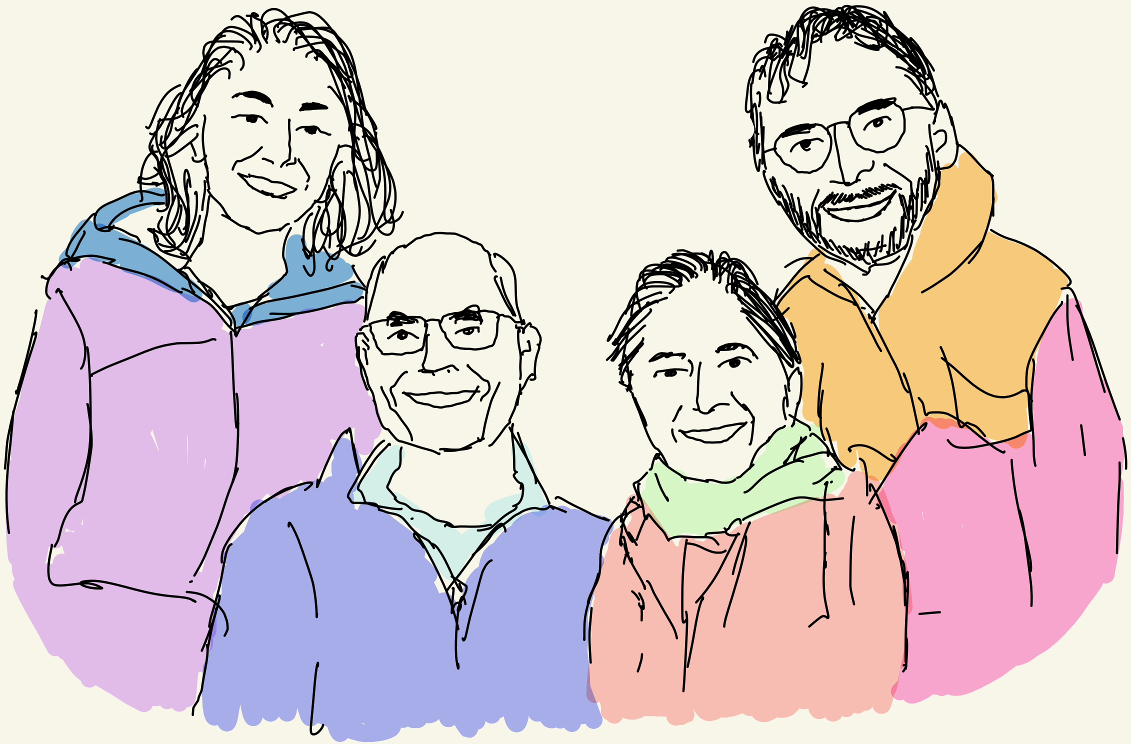
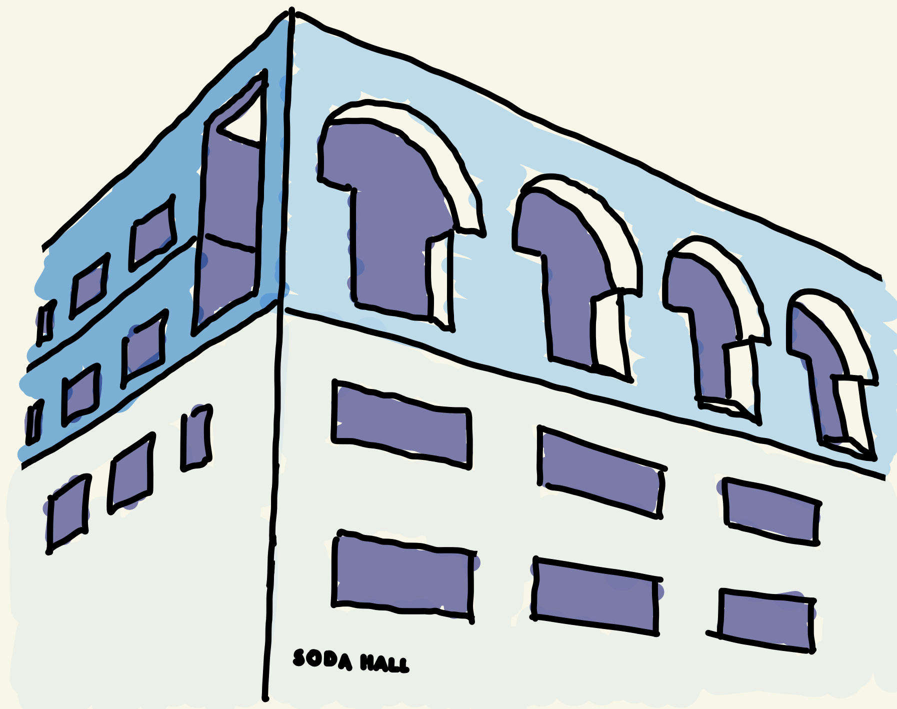
Acknowledgments: My wonderful family



Acknowledgments: The best research environment



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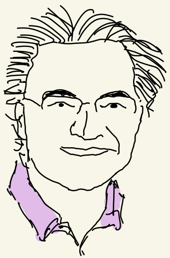

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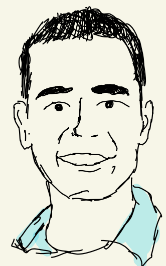

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