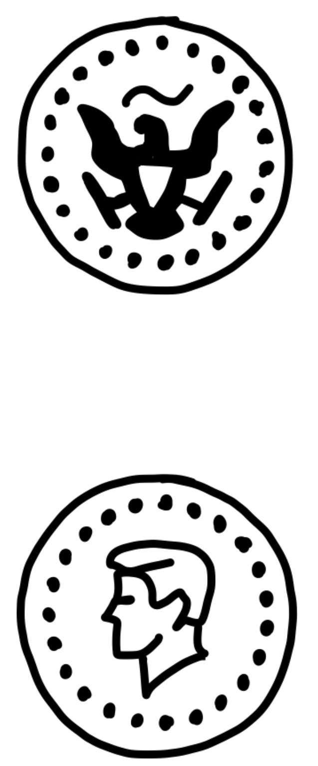
Why is quantum mechanics so hard to describe?

A computational lens on the problem

Chinmay Nirkhe | University of Washington

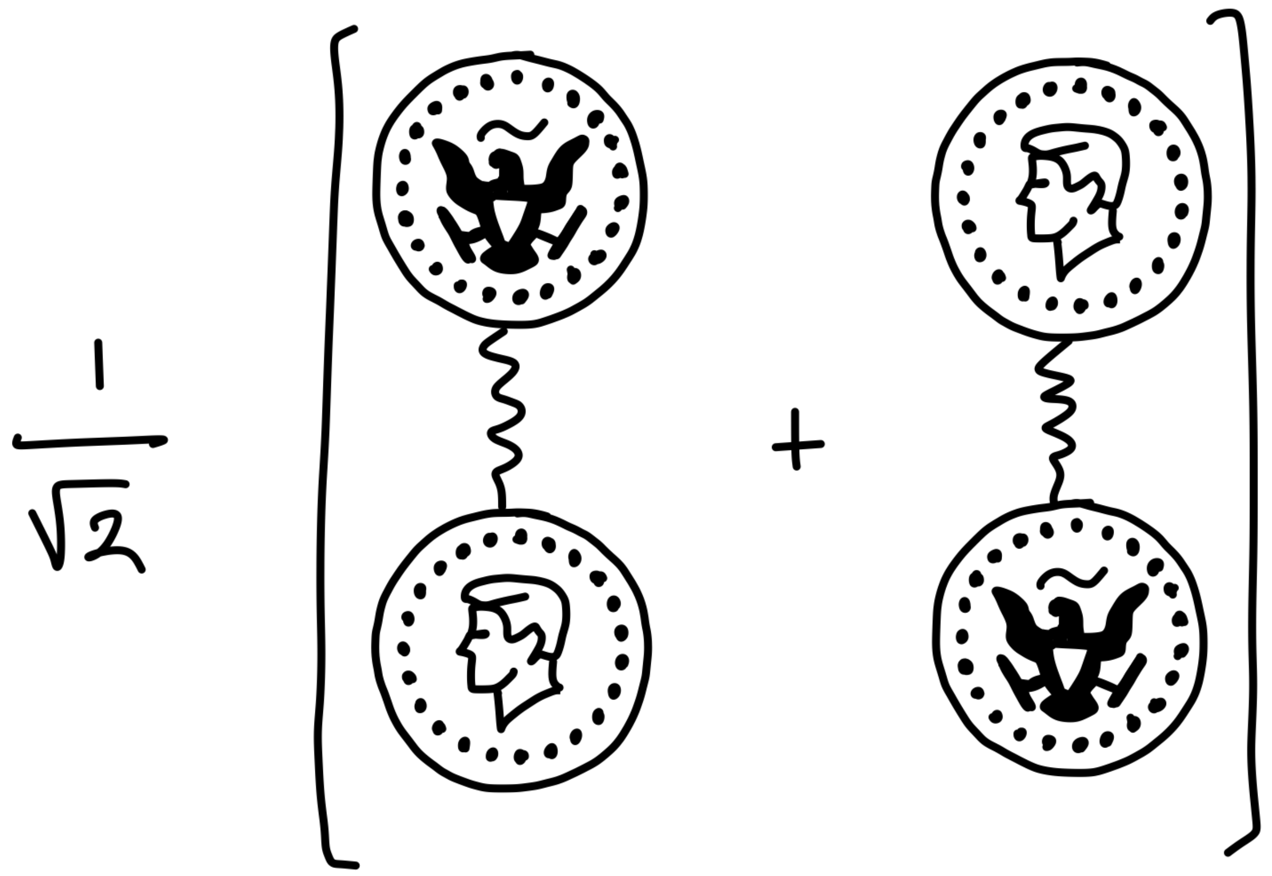
Quantum states are exponentially complex

a classical system



$$|0\rangle|1\rangle$$

a quantum system



$$\frac{|0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{2}}$$

quantum entanglement

doesn't suffice to describe each individual particle

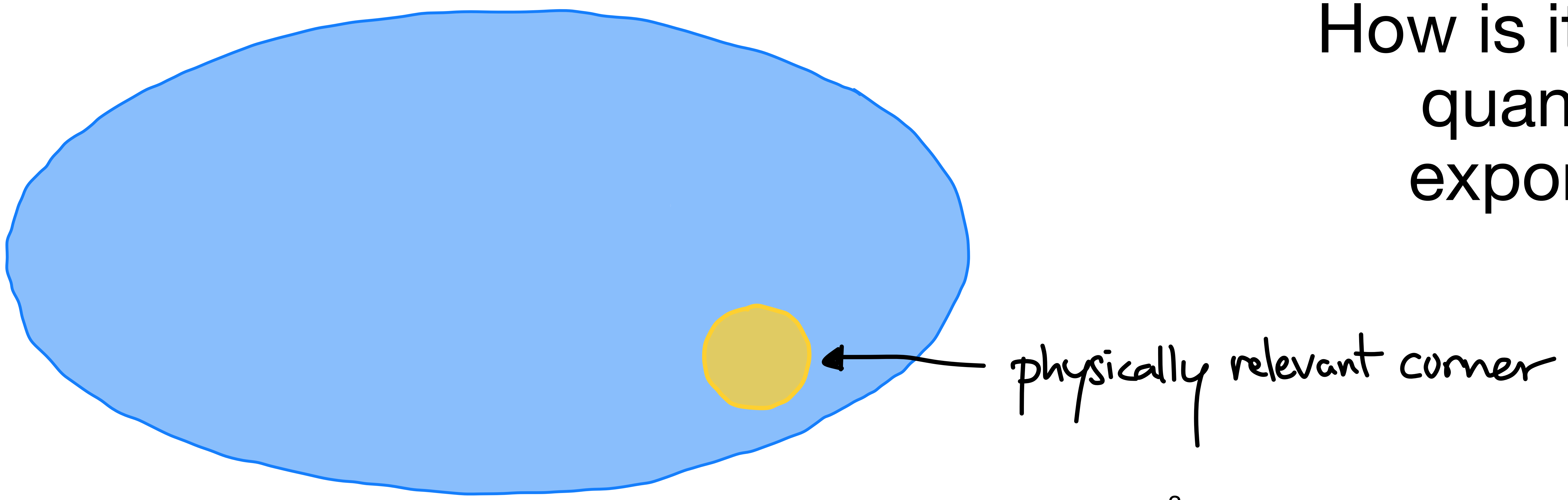
Quantum states are exponentially complex

n particles (each with 2 states) ○ ○ ○ ○ . . . ○

$$\text{quantum state} = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle = \begin{pmatrix} \alpha_{000} \\ \alpha_{001} \\ \alpha_{010} \\ \vdots \\ \alpha_{11\ldots1} \end{pmatrix} \quad 2^n$$

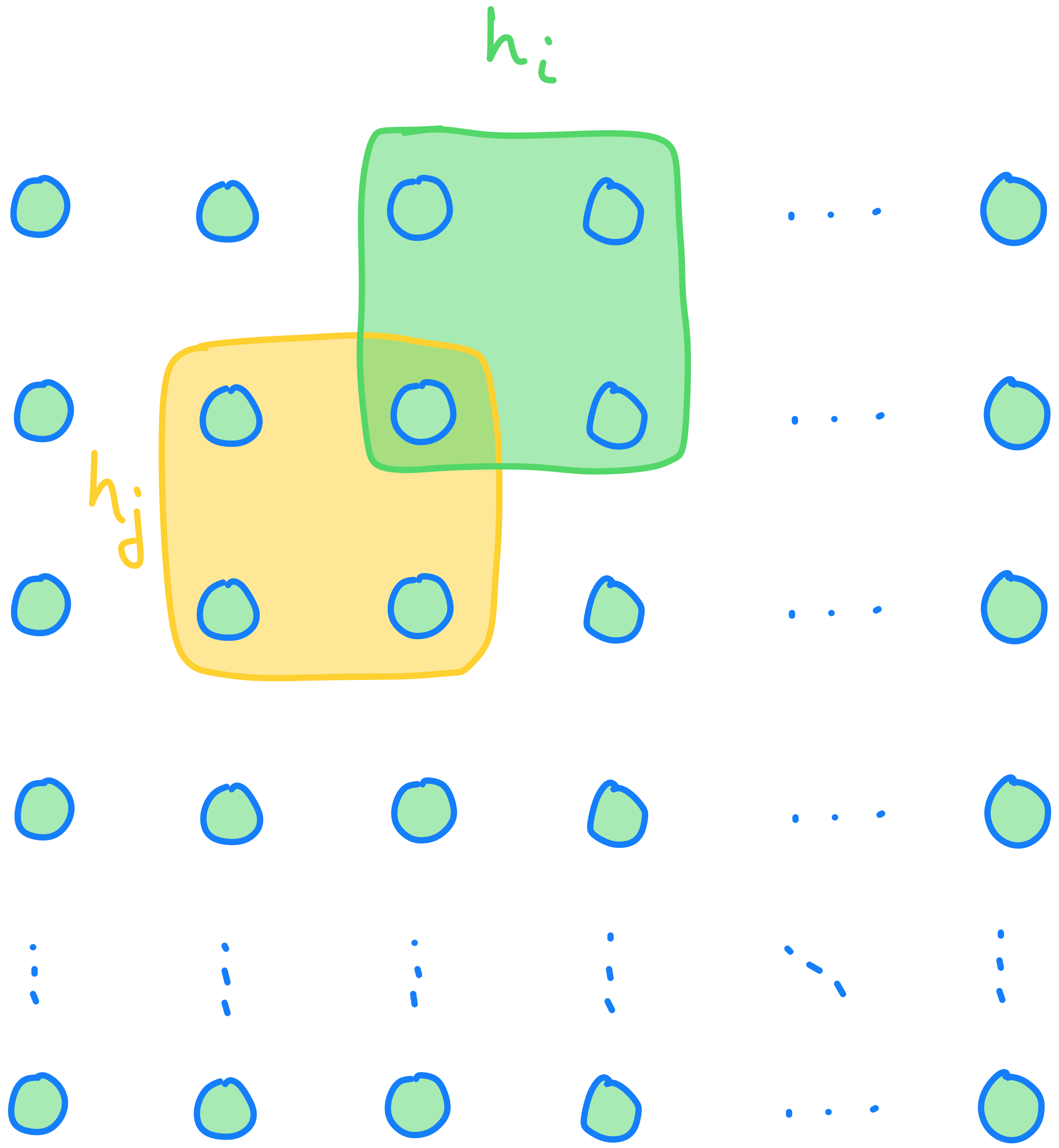
Sum over all n-bit strings

space of n particle states $= \mathbb{C}^{2^n}$



How is it possible to represent quantum states without exponential complexity?

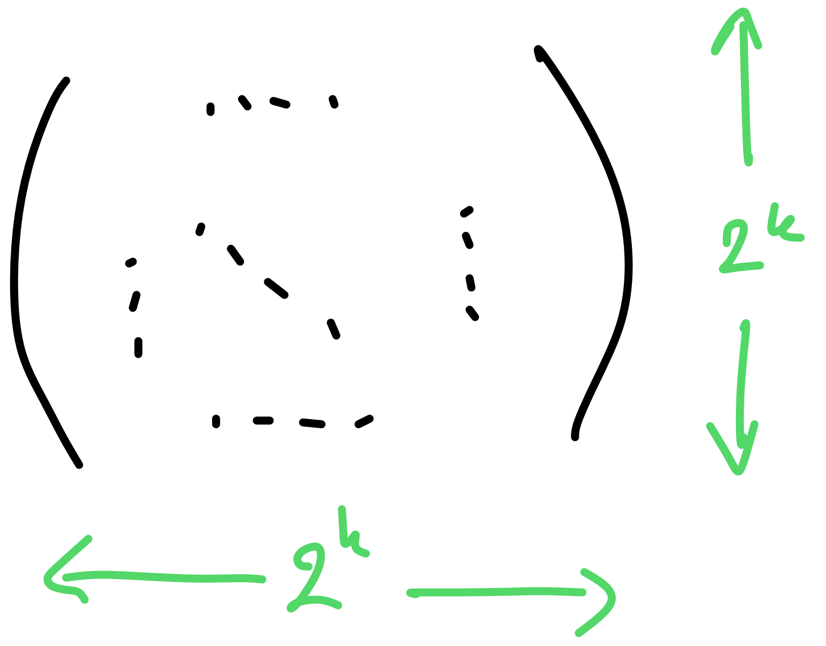
The physically relevant corner



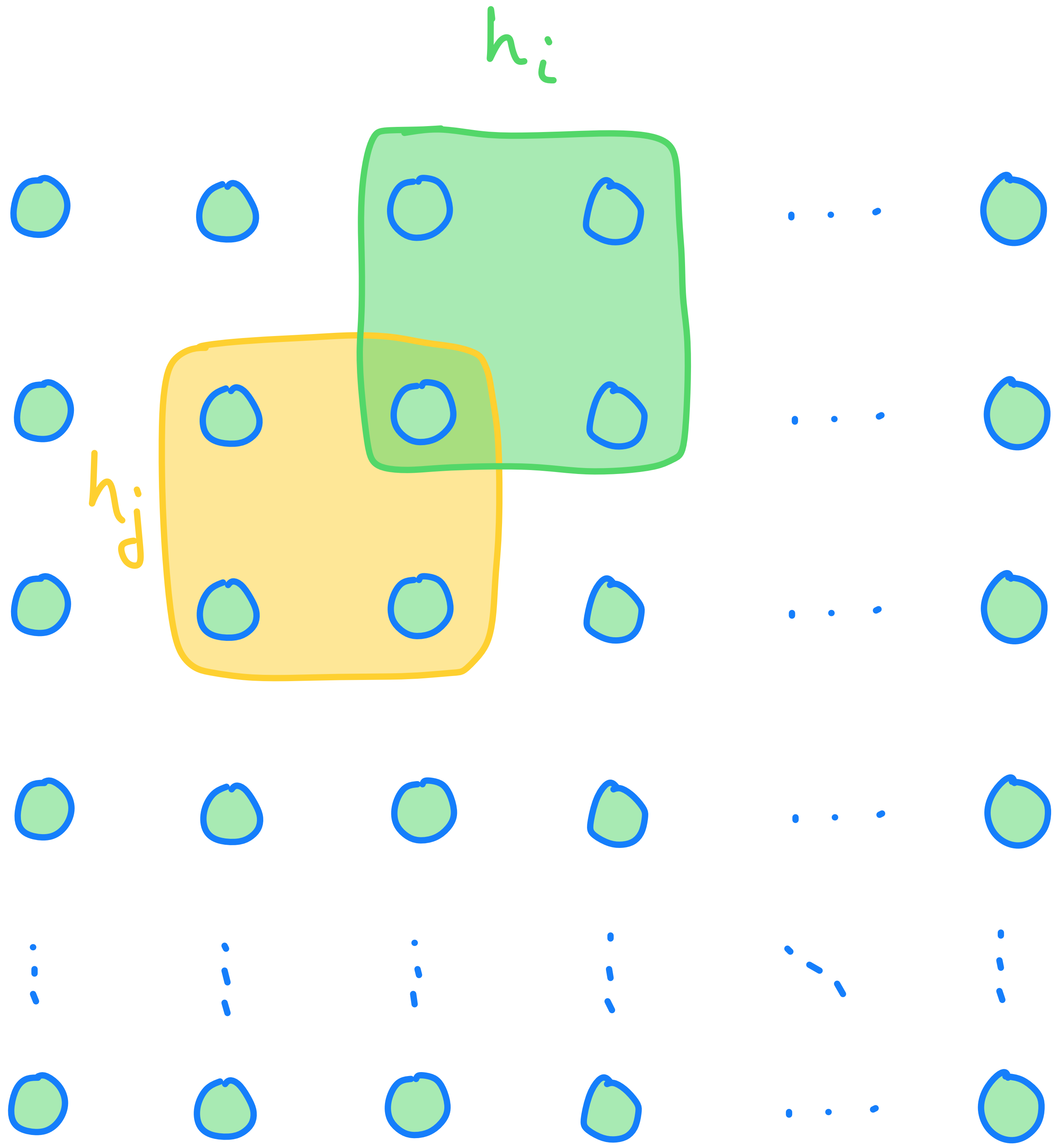
defined by local interactions

each k - local interaction is described by

Hamiltonian h_i =



The physically relevant corner



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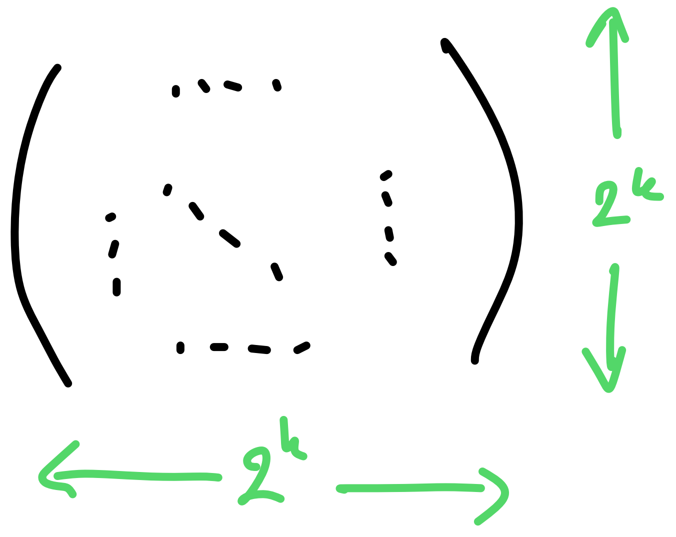

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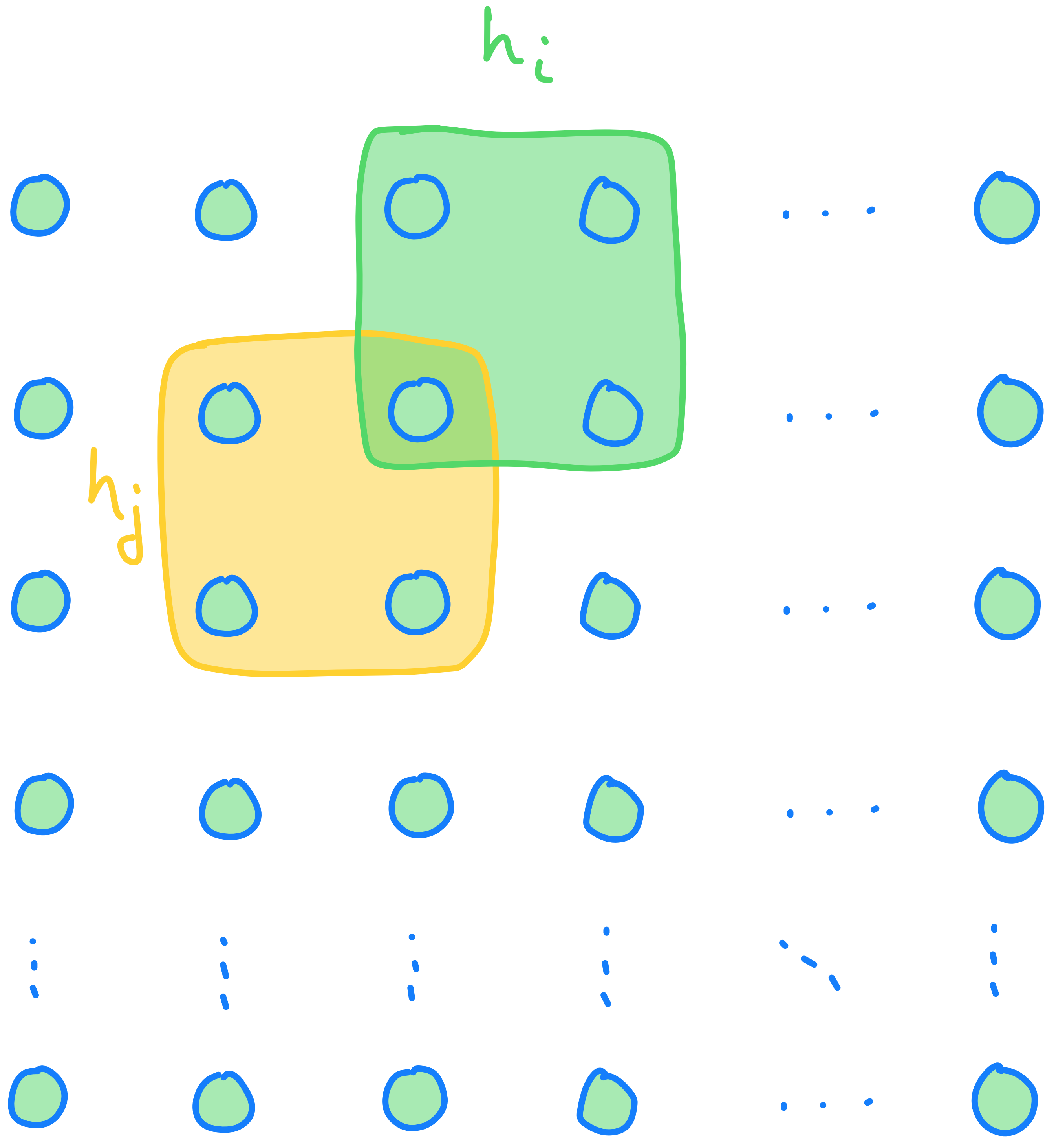

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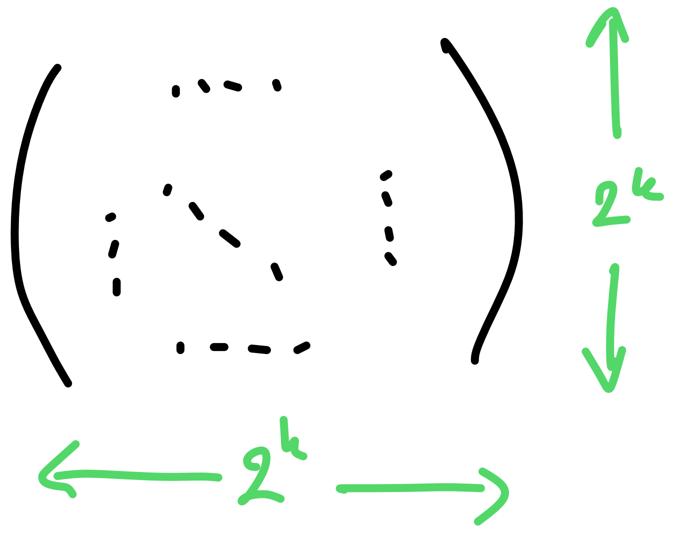
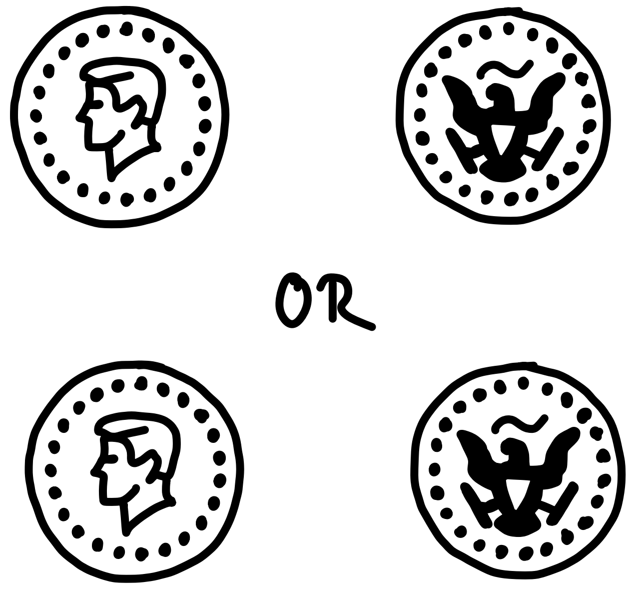
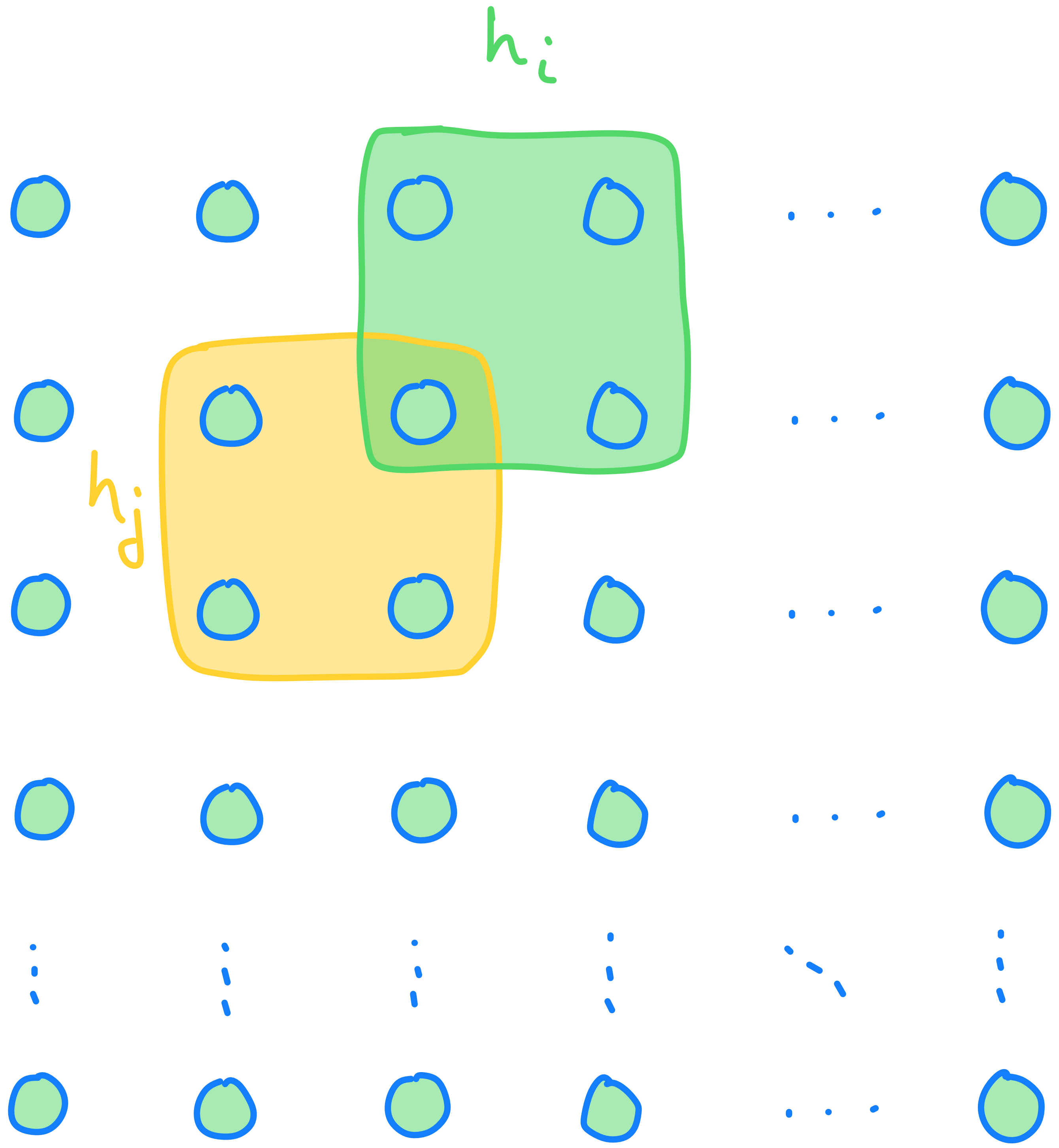


example h_i prefers



$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The physically relevant corner



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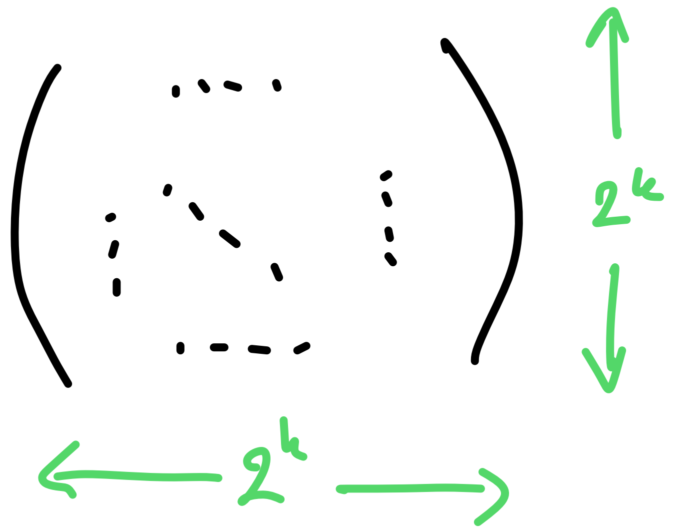

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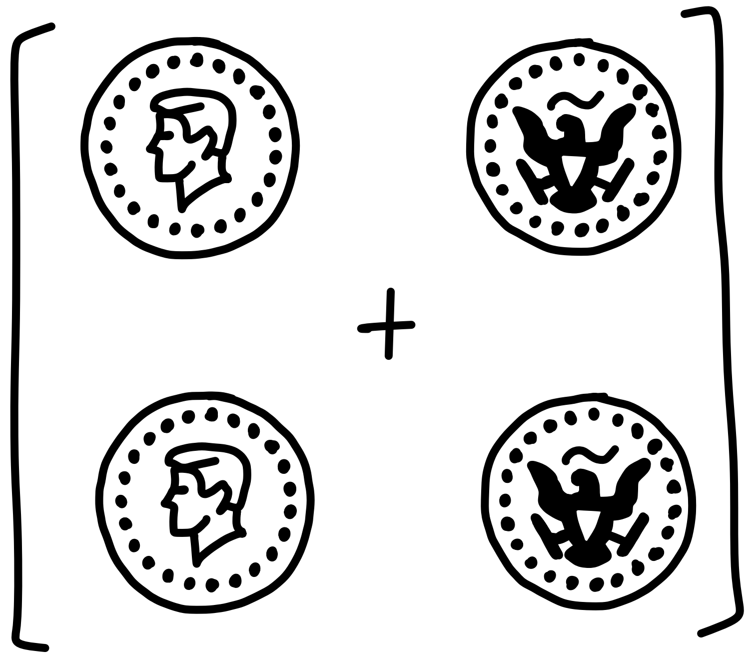
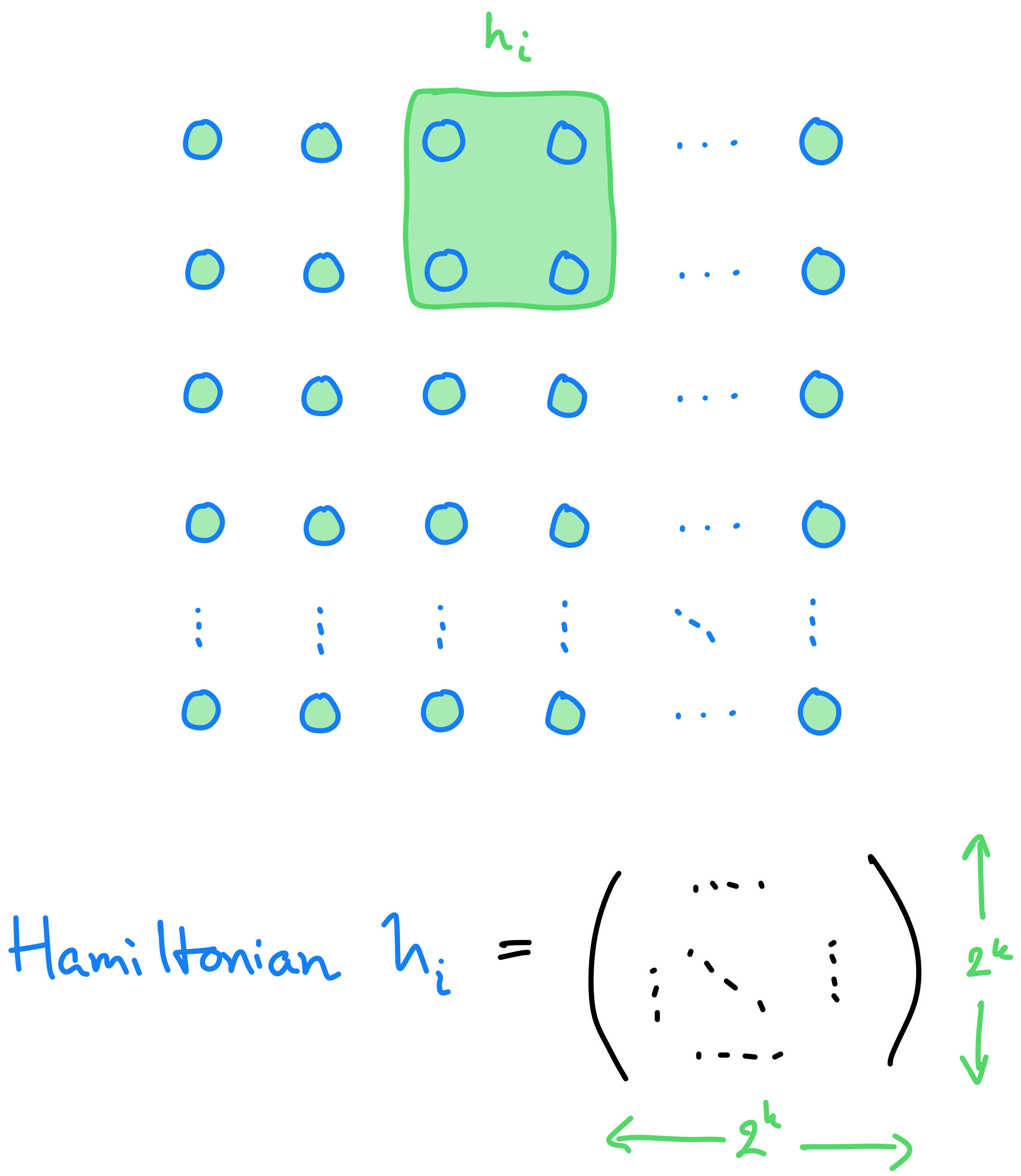
example h_i prefers $\frac{1}{\sqrt{2}}$

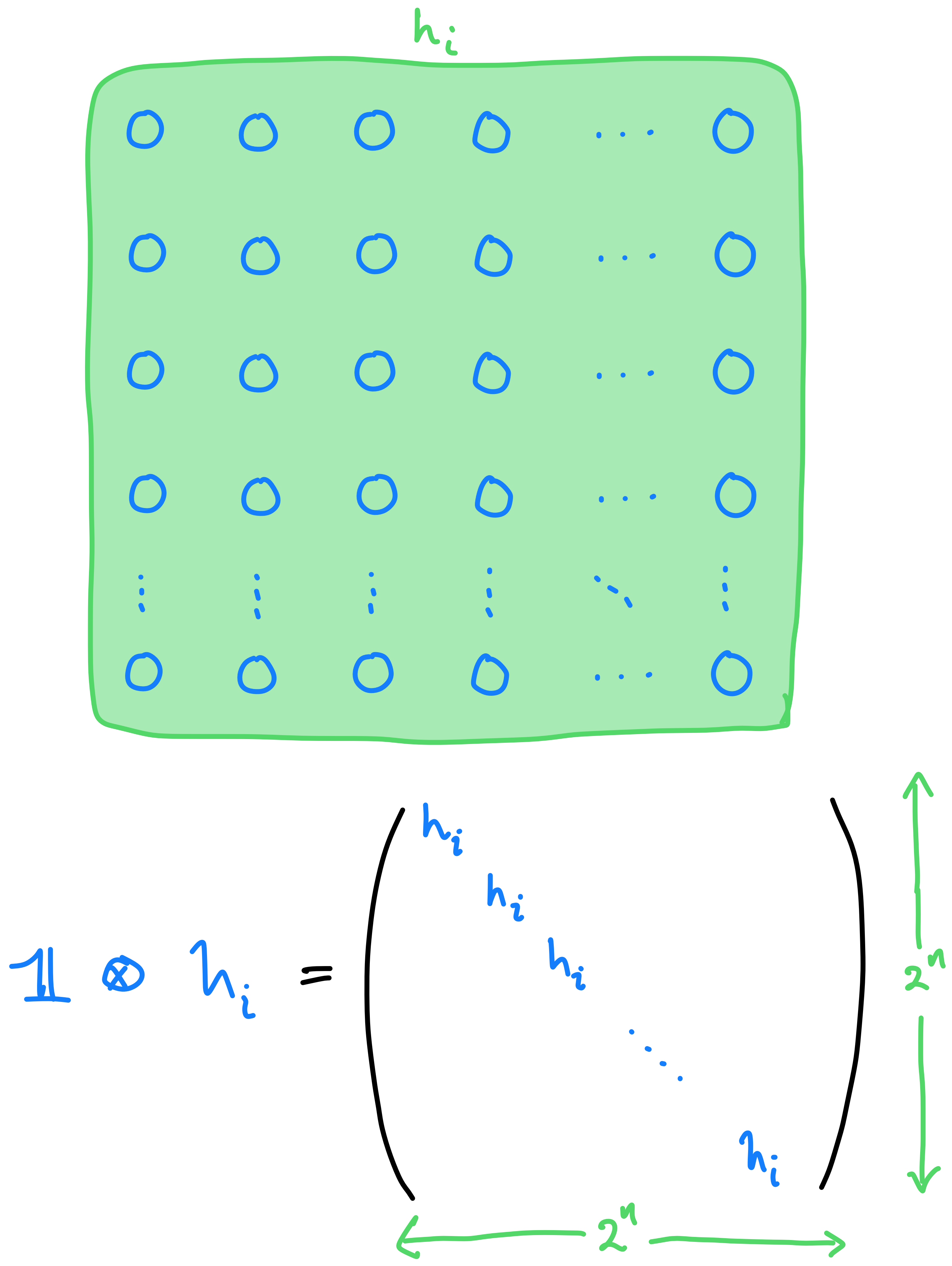


$$\begin{pmatrix} -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

A global view on interactions

Local → Global Phenomenon





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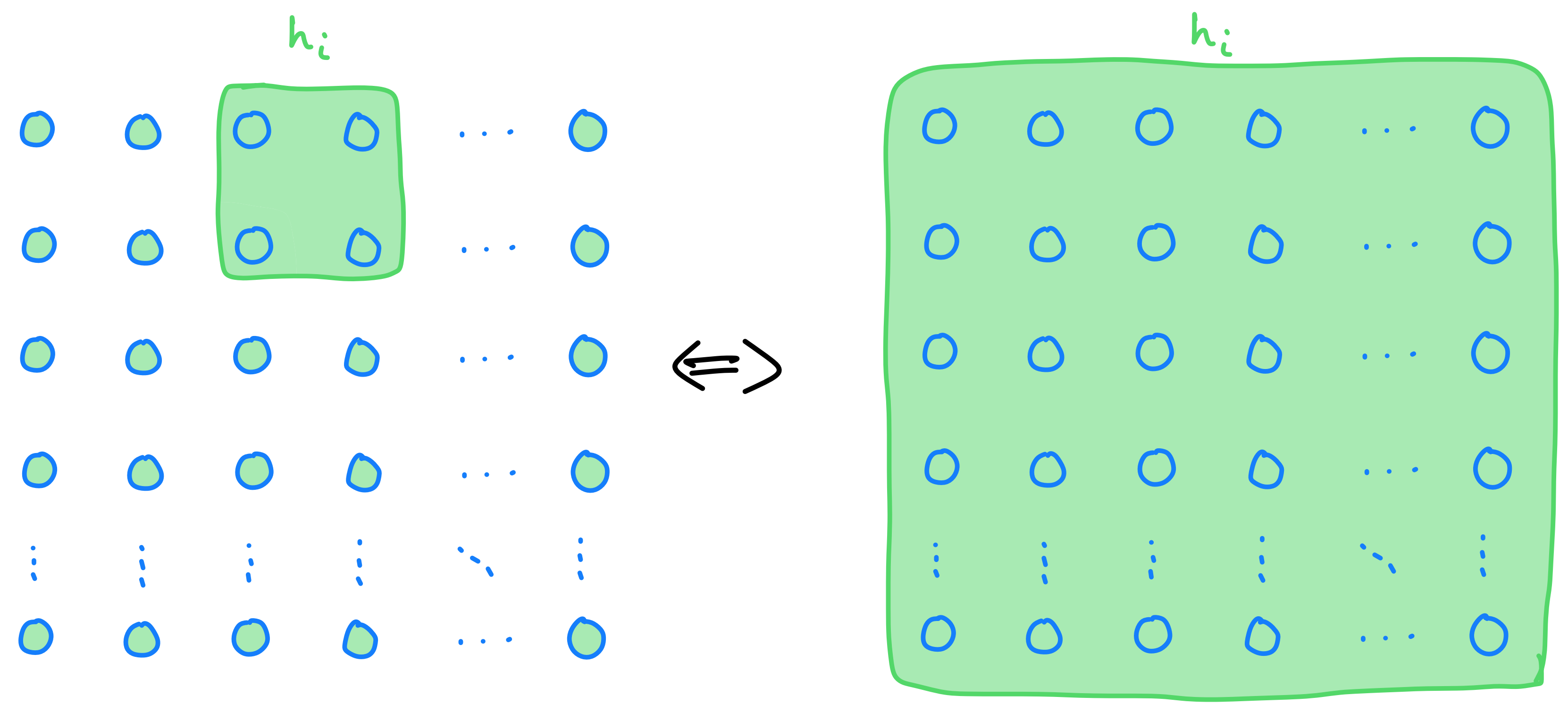
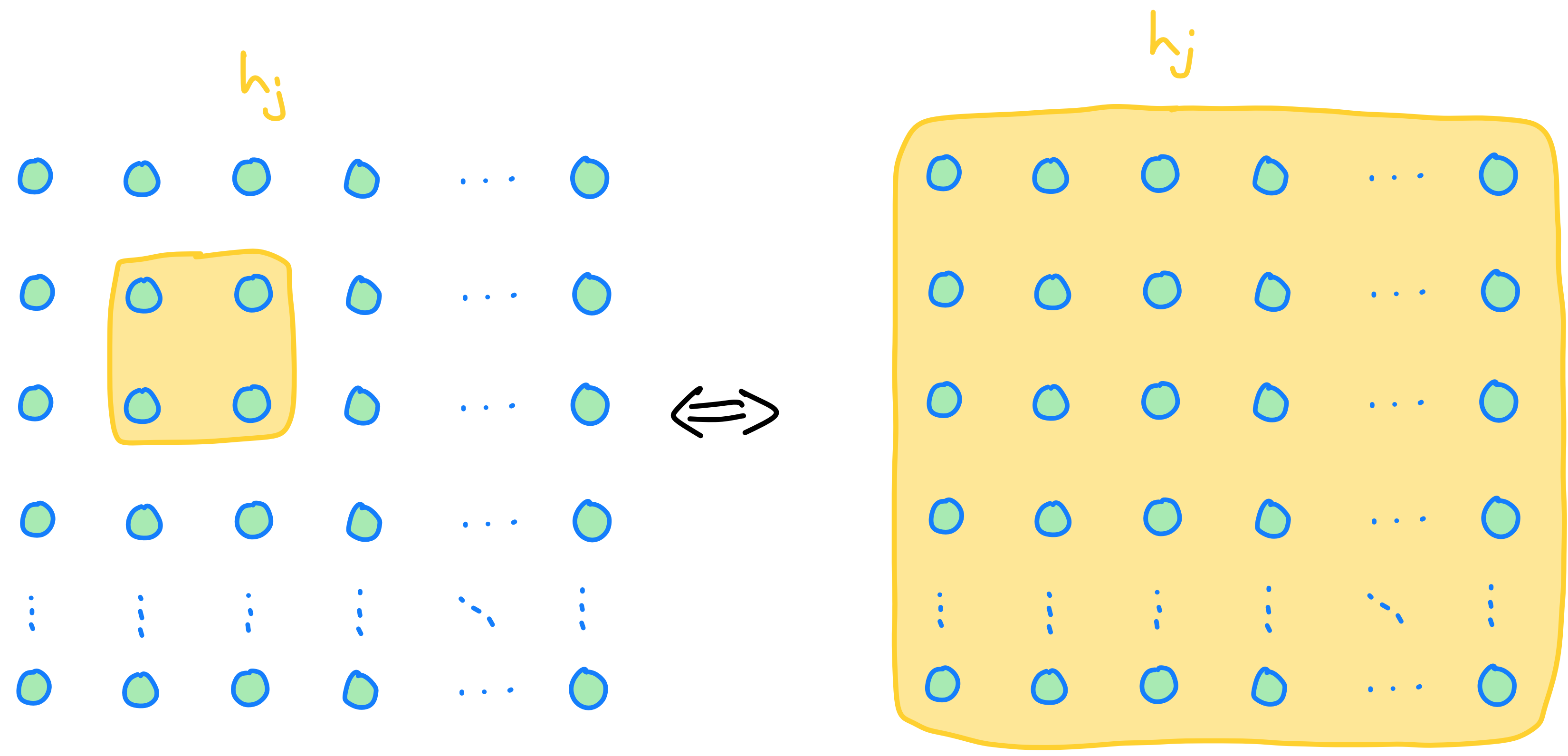




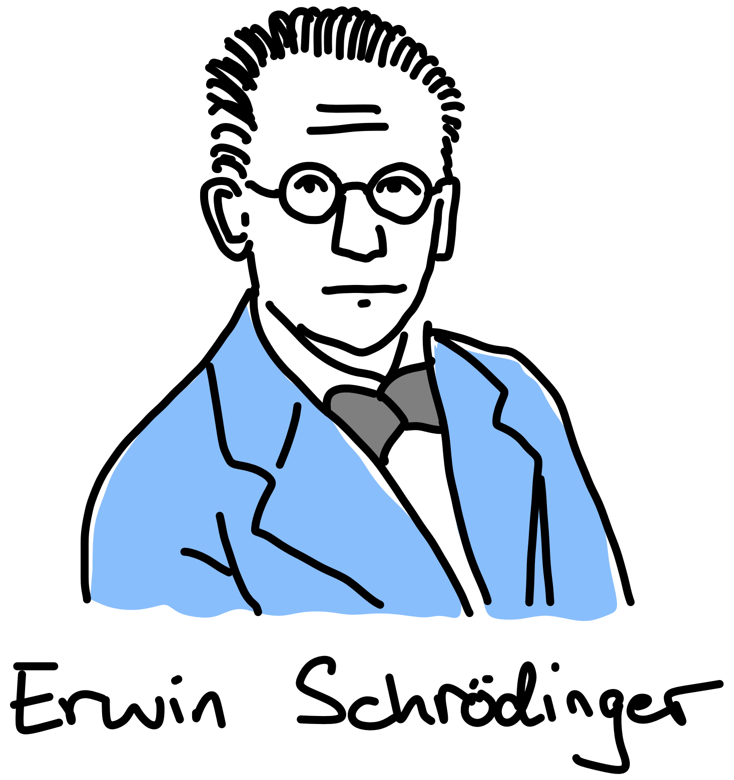
Each local interaction can be viewed globally as a matrix $h_i \in \mathbb{C}^{2^n \times 2^n}$.

Together $H = \sum_i h_i$

is called the local Hamiltonian and is the matrix governing physical interactions

A global view on interactions

Local → Global Phenomenon

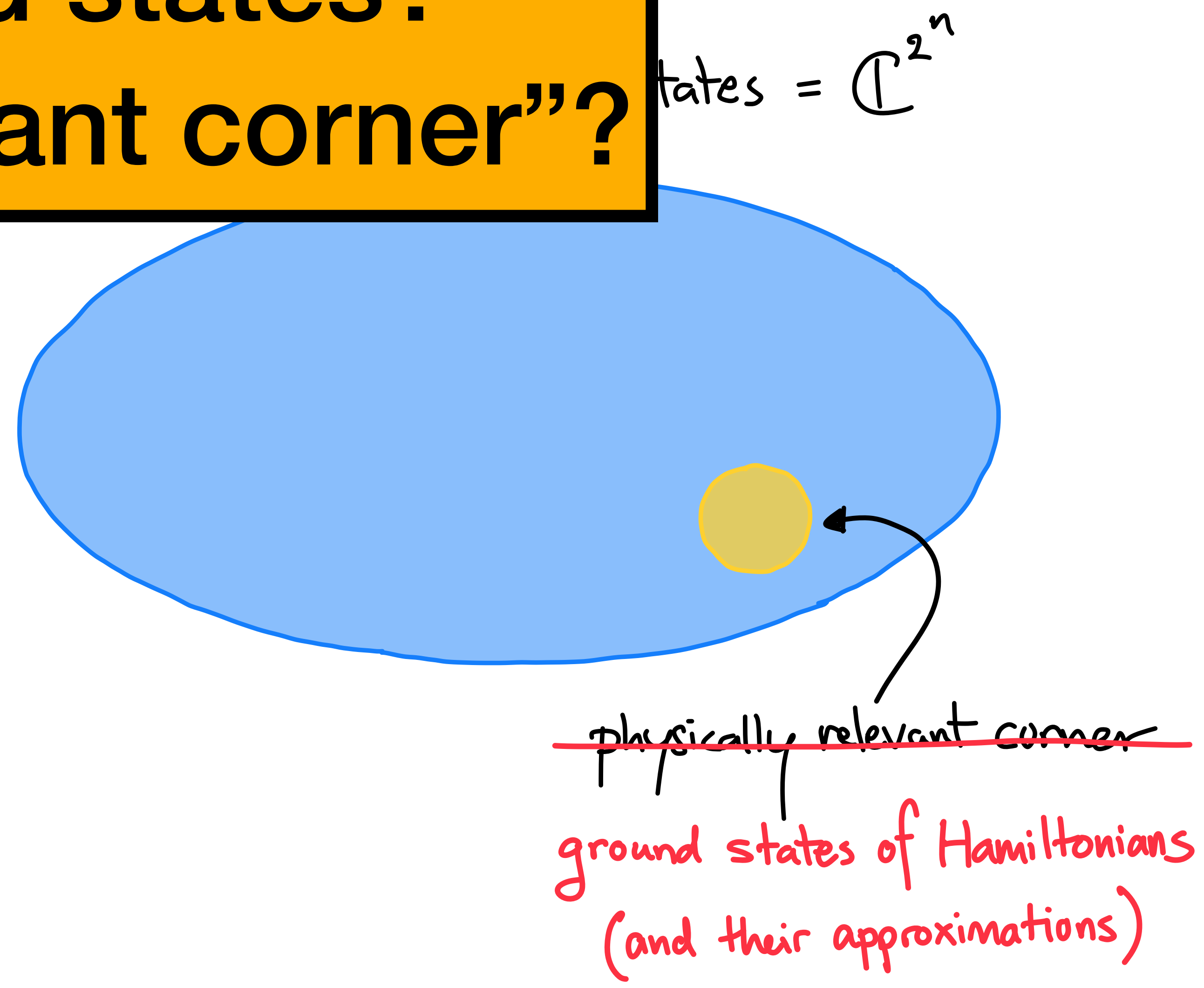


- Physical systems are governed by local Hamiltonians $\mathbf{H}$
- Schrödinger equation:

$$\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t}$$

Do we still need $\exp(n)$ many bits to describe ground states?
Or is there a better way to describe the "physically relevant corner"?

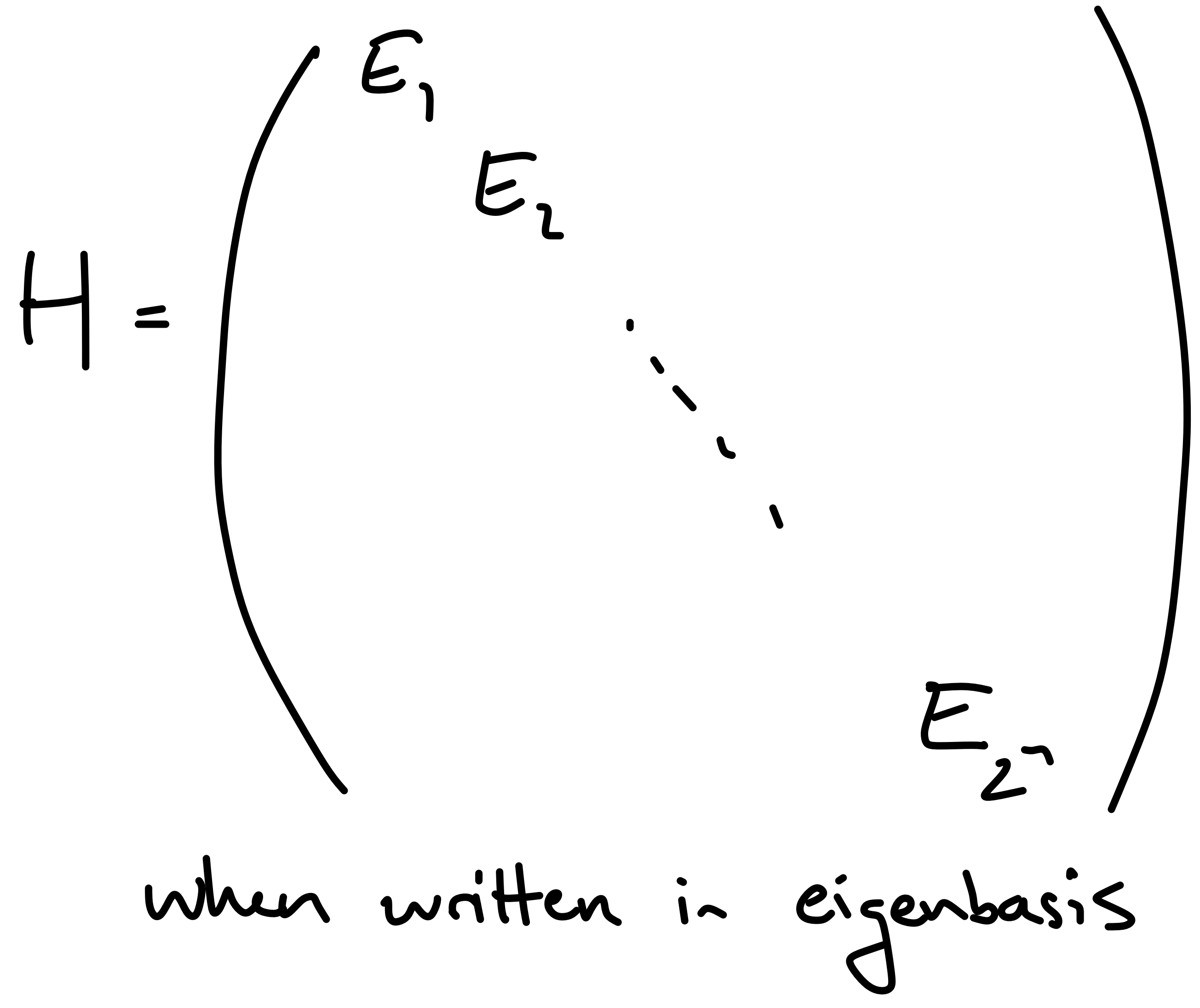
- The re
  These are the stable solutions: $\mathbf{H}|\psi\rangle = E_\psi|\psi\rangle \implies |\psi(t)\rangle = e^{\hat{i}E_i t}|\psi(0)\rangle$



- In particular, physics is interested in understanding the **ground states of $\mathbf{H}$** (and any low-energy approximations)
- **Only a tiny fraction of all states** are also ground states of local Hamiltonians. This gives a formal definition to the "physically relevant corner".

A global view on interactions

A brief primer on ground states



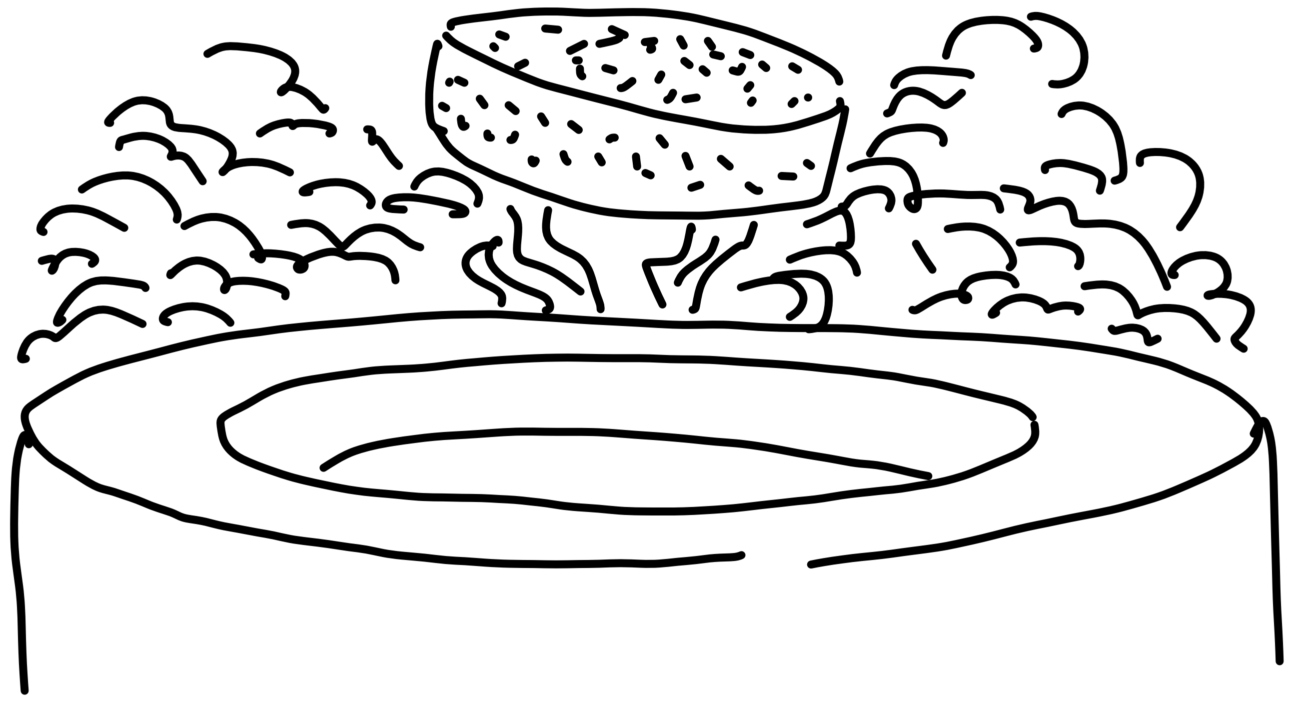
- The local Hamiltonian $\mathbf{H}$ is a Hermitian matrix ($\mathbf{H} = \mathbf{H}^\dagger$) as it is the sum of local interactions each describable by a Hermitian matrix
- Every Hermitian matrix can be *diagonalized* into its eigenbasis
- In the bra-ket physics notation, we write this as $\mathbf{H} = \sum_{i=1}^{2^n} E_i |\psi_i\rangle\langle\psi_i|$

  eigenvalue
  eigenvector
- The ground energy $E_{\min}$ is $:= \min_i E_i$ and the ground states are any corresponding vectors/states
  - Equivalently, $|\psi_{\min}\rangle$ is a ground state iff $\mathbf{H}|\psi_{\min}\rangle = E_{\min}|\psi_{\min}\rangle \iff \langle\psi_{\min}|\mathbf{H}|\psi_{\min}\rangle = E_{\min}$.
  - Mathematically, $E_{\min}$ is the minimum eigenvalue of $\mathbf{H}$ and $|\psi_{\min}\rangle$ is the partner eigenvector.

The central problem in condensed matter physics

Given the description of a local Hamiltonian system $\mathbf{H}$, calculate a description of the ground state as well as calculate properties of the ground state.

- Spectral gap of the Hamiltonian
- Correlations between different particles
- Entropy structure of ground states and complexity of many-body entanglement
- Topological symmetries / phases of matter
- Superfluidity/superconductivity and other phenomena



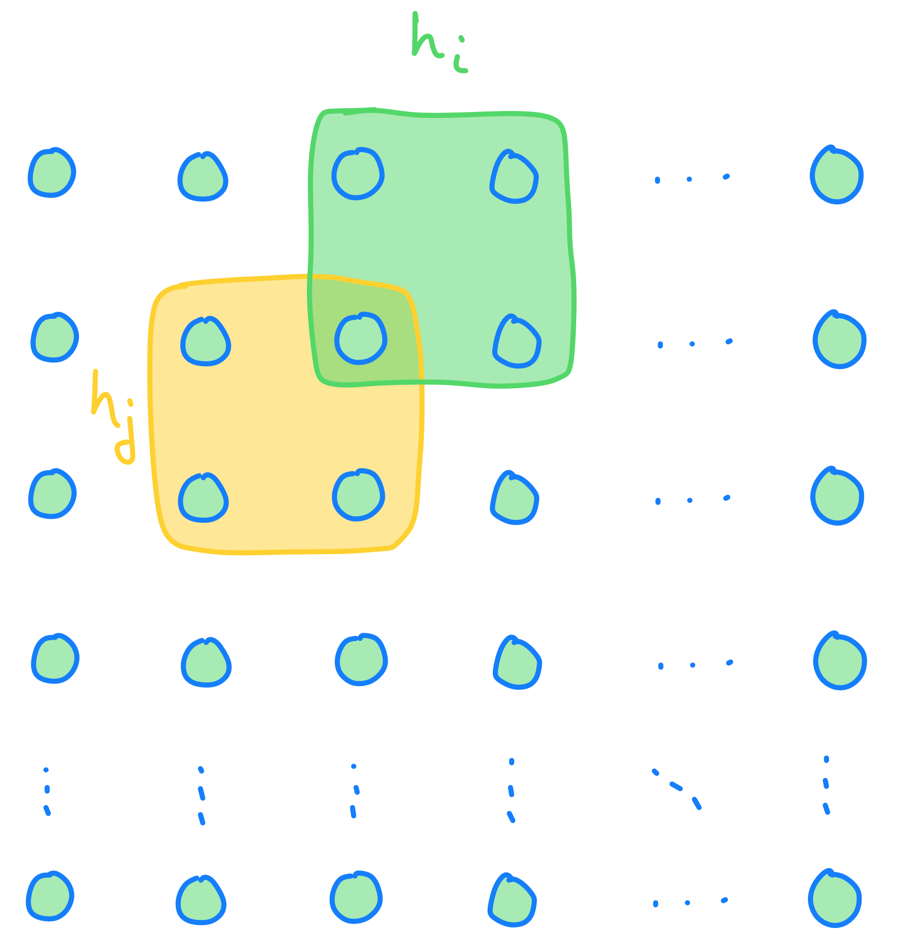
The central problem in condensed matter physics

Given the description of a local Hamiltonian system $\mathbf{H}$, calculate a description of the ground state as well as calculate properties of the ground state.

Calculating ground states of n-qubit systems involves "stitching" together local solutions of the individual local terms $h_i \in \mathbb{C}^{2^k \times 2^k}$ for $k = O(1)$

The complexity of ground states

- Let's start with an "easier" question.
- **Input:** The classical description of a local Hamiltonian $\mathbf{H}$
  - An m-term and k-local Hamiltonian system can be classically described in $O(m \cdot 2^{2k})$ bits
  - When $m = \text{poly}(n)$ and $k = O(1)$, this is $\text{poly}(n)$
- **Output:** The classical value of the ground energy: $E_{\min}$.
- **First question:** Why is this the right computational task to consider first? What do algorithms for this task tell us?



$$\Downarrow$$

$$E_{min} = \lambda_{min}(H) = \lambda_{min}\left(\sum_i h_i\right)$$

The complexity of ground energy

- **Input:** The classical description of a local Hamiltonian $\mathbf{H}$.
  Output: The classical value of the ground energy: $E_{\min}$.
- This is an example of a classical input $\rightarrow$ classical output problem
- There is a naive classical algorithm for it as well!
  - Since $\mathbf{H}$ is a $2^n \times 2^n$ matrix, we can calculate $E_{\min}$ (on a classical computer) by diagonalizing.
  - This algorithm runs in time $\exp(n)$. Is there a better algorithm (perhaps a quantum algorithm)?

The complexity of ground energy

- **Input:** The classical description of a local Hamiltonian $\mathbf{H}$.
  Output: The classical value of the ground energy: E_{min}.

- If there was a quantum algorithm which given $\mathbf{H}$ can *construct* the ground state, $|\psi_{\text{min}}\rangle$, then there is a quantum algorithm for computing E_{min}.

- **Proof:** There is a well-known algorithm (phase estimation) for estimating (to arbitrary accuracy) $\langle\psi_{\text{min}}|\mathbf{H}|\psi_{\text{min}}\rangle$ (which equals E_{min}).

- **Corollary:** A *lower bound* on the complexity of computing ground energy implies a *lower bound* on the complexity of generating ground states.

The complexity of ground energy

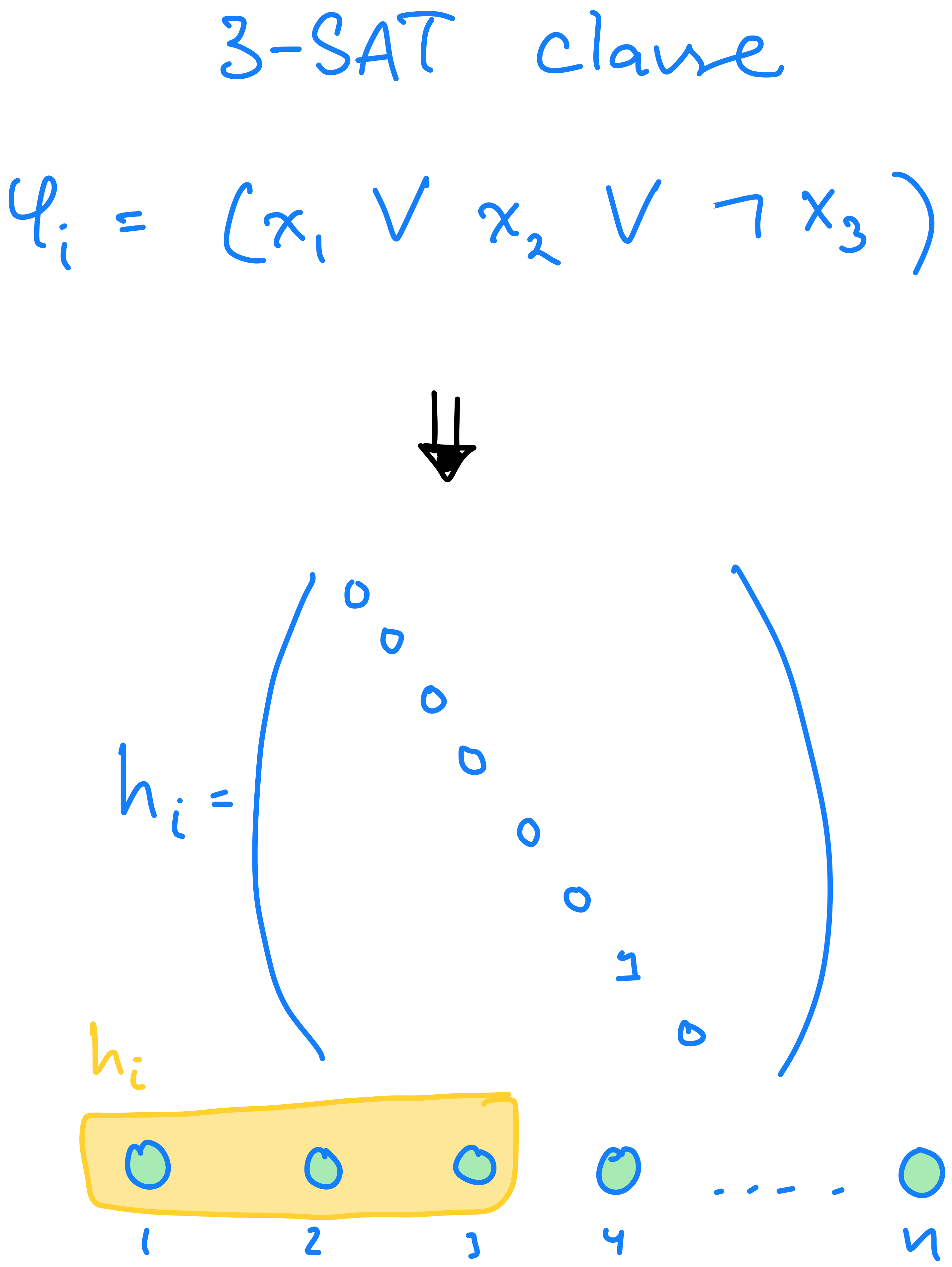
Lower bounds

- **Input:** The classical description of a local Hamiltonian $\mathbf{H}$.
  Output: The classical value of the ground energy: $E_{\min}$.

- If we consider $\mathbf{H}$ where every local term h_i is a diagonal matrix, then this is a generalization of the 3-SAT problem from classical computer science. Famously NP-complete!

  - Any algorithm for computing ground energy also must be able to solve 3-SAT —this is a *lower bound* on the computational complexity.

  - $3 - \mathrm{SAT} \leq_p \mathrm{LH}$



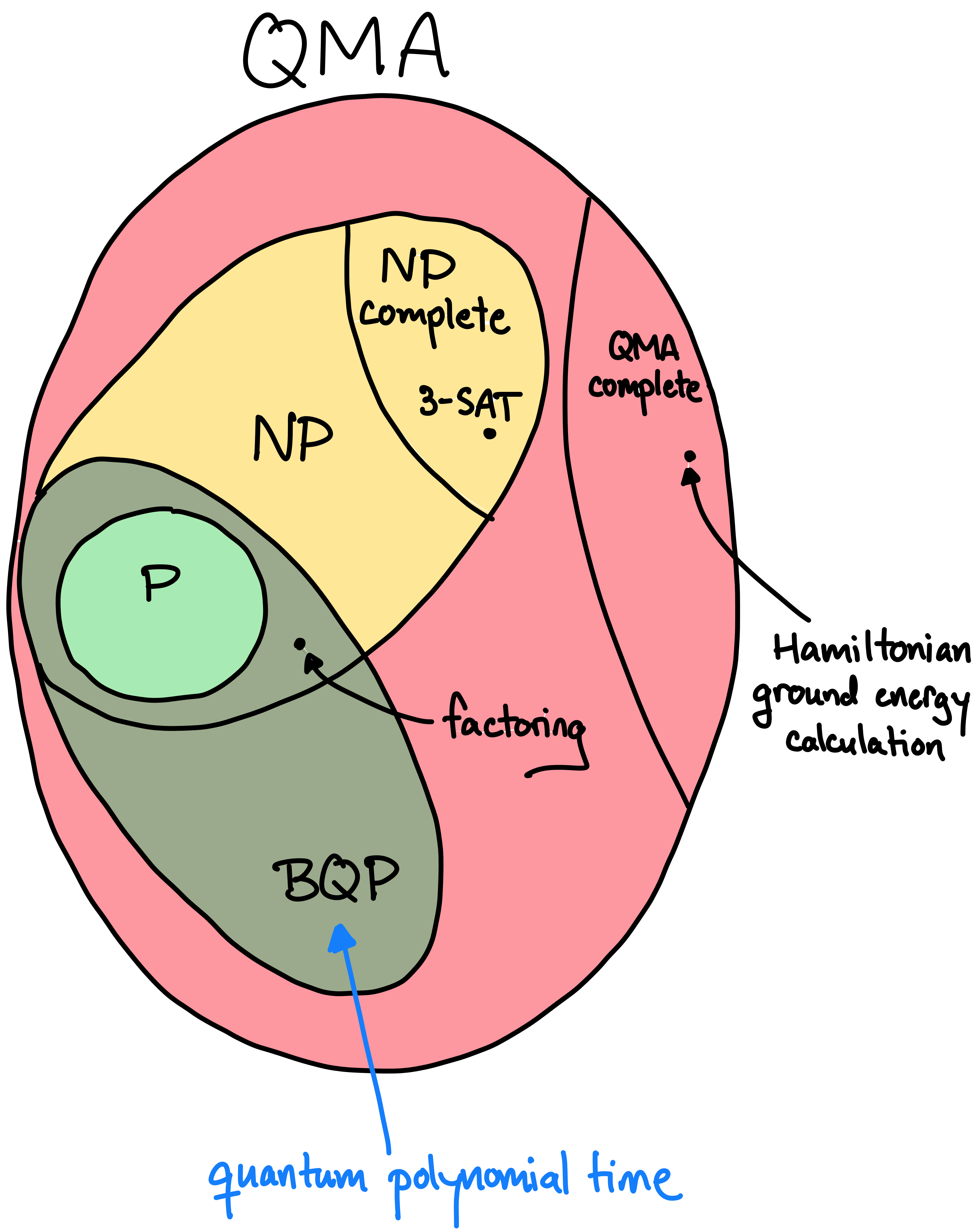
The complexity of ground energy

Tight characterization

- **Input:** The classical description of a local Hamiltonian $\mathbf{H}$.
  Output: The classical value of the ground energy: $E_{\min}$.
- In 2002, Kitaev, proved that any *non-deterministic quantum computation* (quantum analog of NP) can be reduced to calculating ground energy.
  - I.e., an algorithm (either quantum or classical) for computing ground energy would imply an algorithm for any problem that is checkable with a "quantum proof"
  - In particular, proves a distinct *lower bound (i.e., hardness result)* for calculating ground states or their descriptions
  - This proves that the central problem of condensed matter physics is very hard!

Exactly how hard?

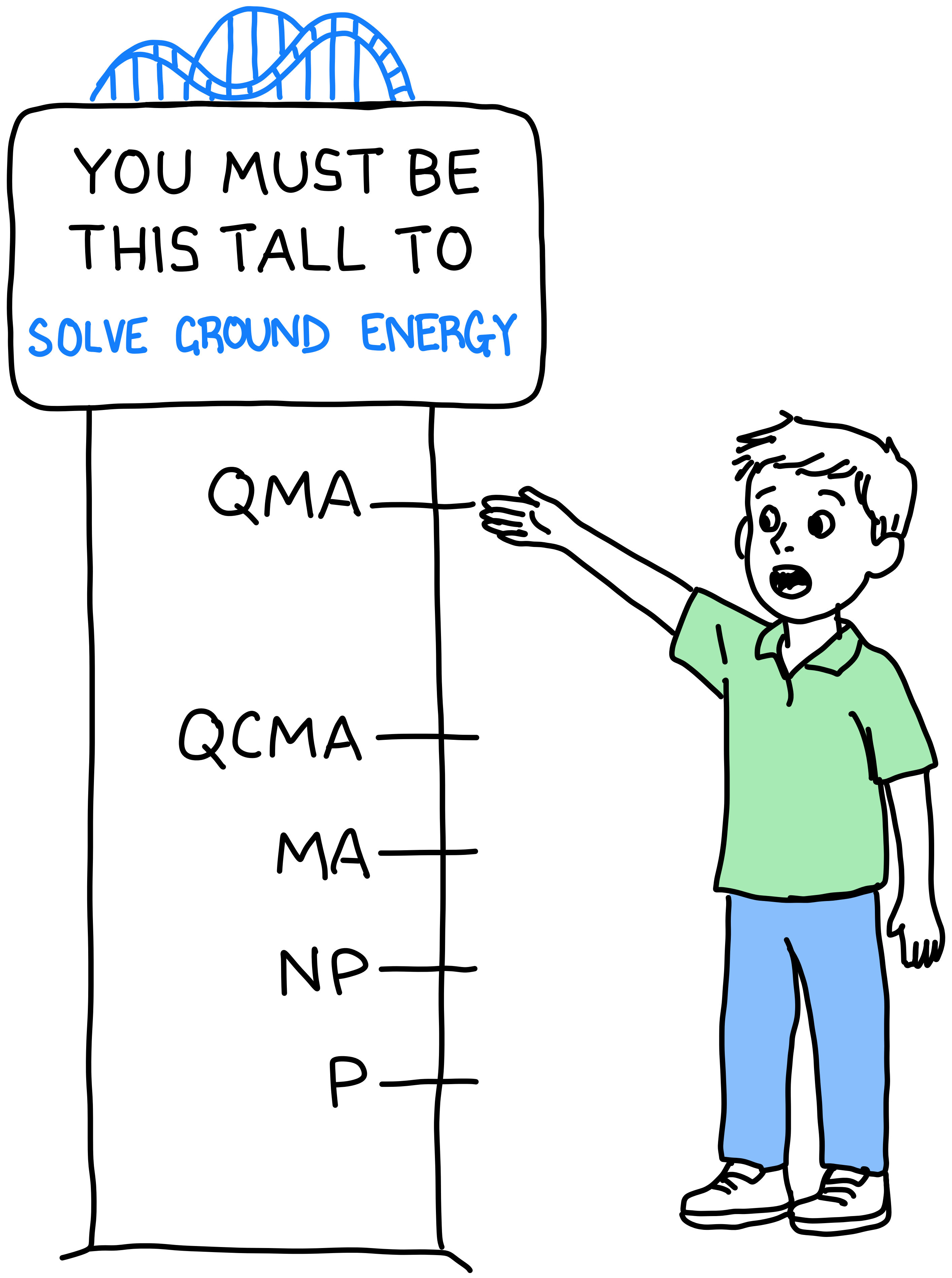
- Kitaev proved calculating ground energy is intractable for quantum computers
- However, as any computer scientist will tell you, intractability is relative
- Specifically, Kitaev proved that calculating ground energy exactly is QMA – complete
    - QMA stands for "Quantum Merlin-Arthur"
    - It is a quantum analog of non-deterministic polynomial time (NP)
- Two natural follow up questions have been driving forces of my research:
    - **Is there an easier (classical or quantum) algorithm for approximating the ground energy? Or an impossibility result?**
    - **How hard are ground states to describe and can we describe ground states efficiently (such as a circuit preparing the state)?**



Part II: Complexity theory

To classify the complexity of problems relating to computing ground energy, we need to spend a moment describing complexity classes.

This will give us a good yard stick for measuring the hardness of problems.



NP = "non-deterministic polynomial time"

MA = "Merlin-Arthur"

NP/MA

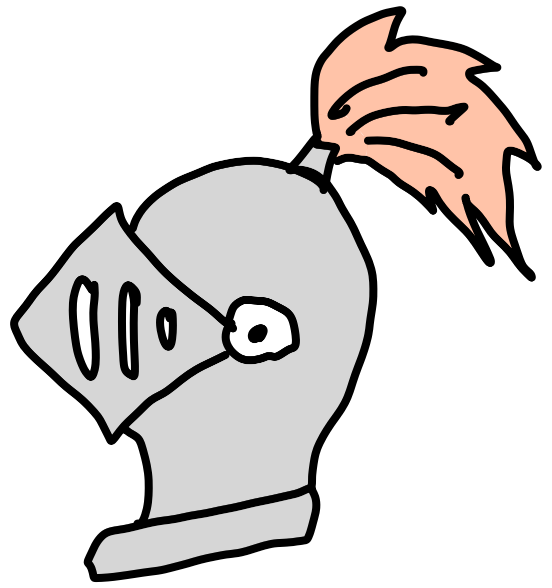
Cook-Levin '71, Karp '72:

3-SAT, 3-Color, Traveling Salesman, Vertex Cover, Hamiltonian Path, et. Are NP-complete.

Merlin (Prover)

classical witness/message w

Is $x \in L$?



Arthur (Verifier)

Arthur is a classical polytime computer and computes a fn $V(x, w)$

There are two appropriate quantum analogs to NP

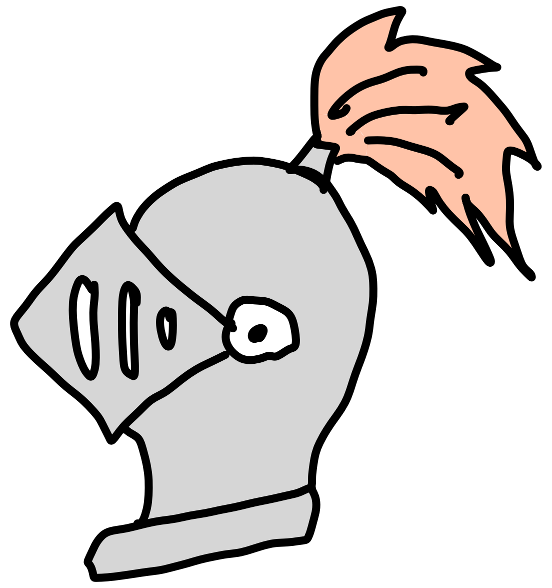
QCMA

quantum classical Merlin Arthur

Merlin (Prover)

classical witness/message w

Is $x \in L$?



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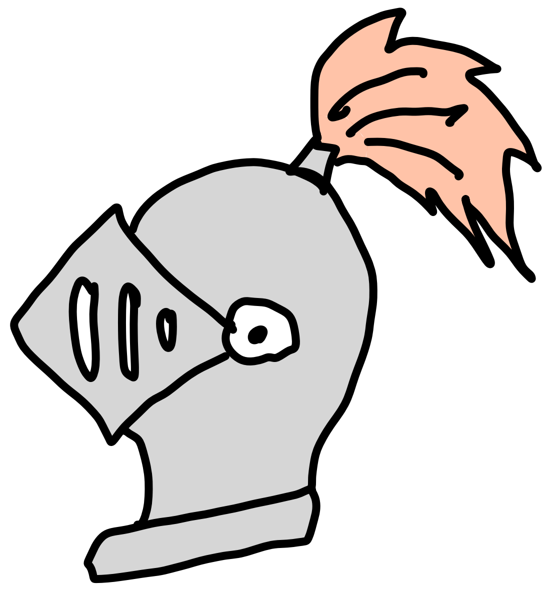
QMA

quantum ~~classical~~ Merlin Arthur

Merlin (Prover)

quantum witness/message $|w\rangle$



Arthur (Verifier)

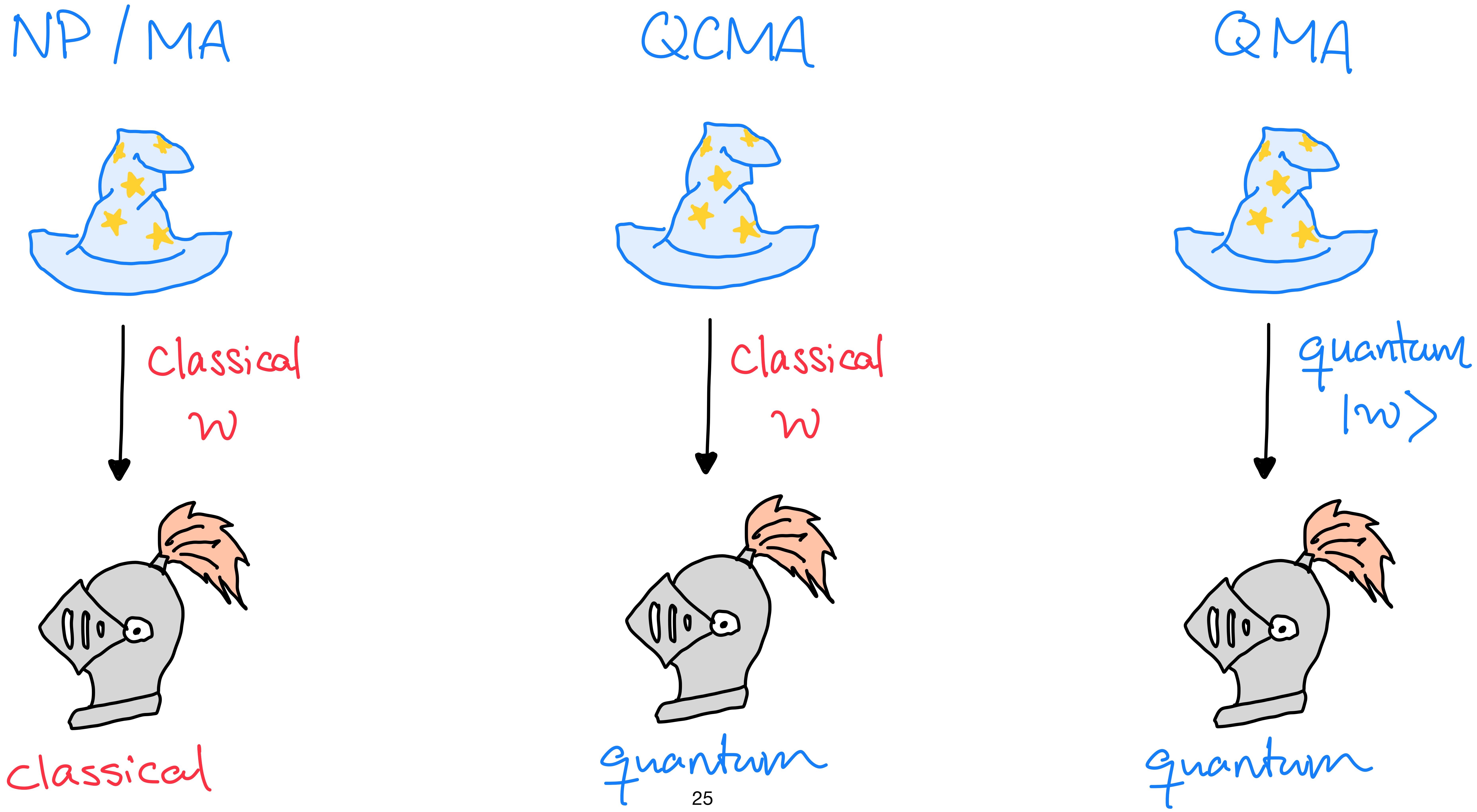
Is $x \in L$?

Kitaev 2002:

Computing ground energy is QMA-complete.

Arthur is a quantum polytime computer and computes a fn $V(x, w)$

There are two appropriate quantum analogs to NP



Why should you care?

Quantum states are hard to describe

Say Arthur wants to understand the ground state $|\psi\rangle$ of a local Hamiltonian system $\mathbf{H}$.

Computing the energy of $|\psi\rangle$ is **QMA**-complete.

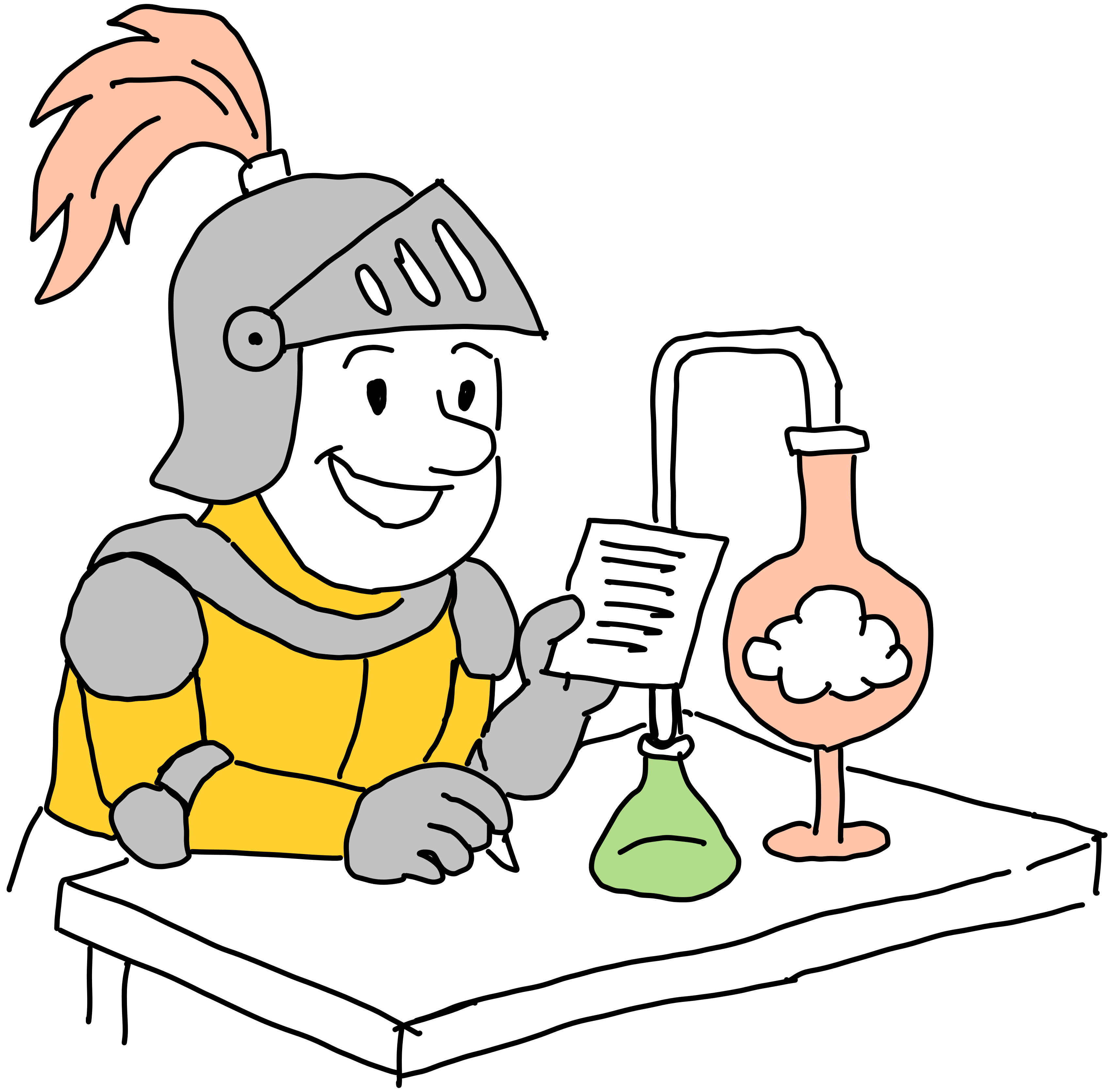
But assuming **BQP≠QCMA**,

Arthur is not able to write down classical descriptions of $|\psi\rangle$ even if he has a copy of $|\psi\rangle$ in his lab.

Why care?

Assuming **QCMA=QMA**,

Merlin can write a classical message about $|\psi\rangle$ which Arthur can use to verify that $\mathbf{H}$ has a low-energy ground state.



Why care?

However, if **QCMA≠QMA**,

Merlin struggles to write a description of $|\psi\rangle$ that Arthur will be able to understand.

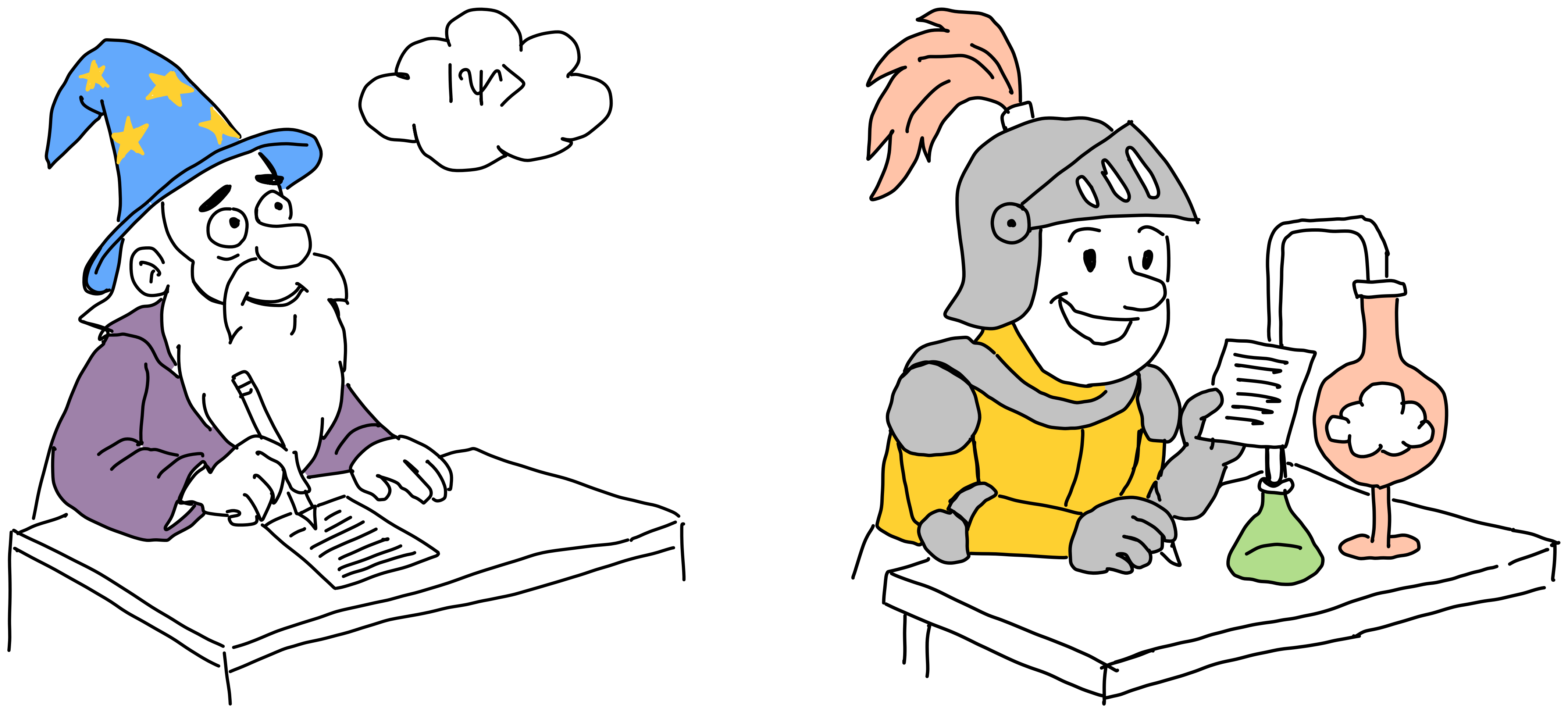
Meaning that $|\psi\rangle$ can be verified but not described.

The only hope for Arthur is if Merlin sends him a copy of $|\psi\rangle$ which we will be able to verify but not write down a classical description of.

Resolving QMA v QCMA

Is deciding which of the two worlds we live in

OR

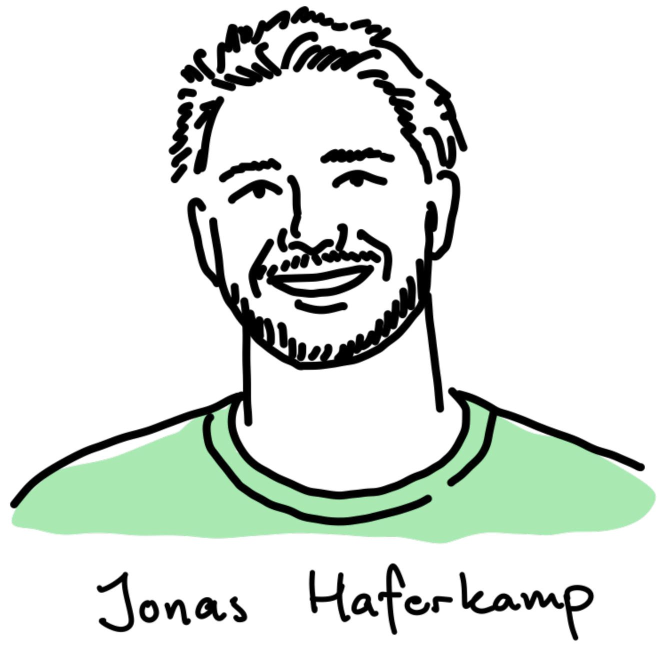


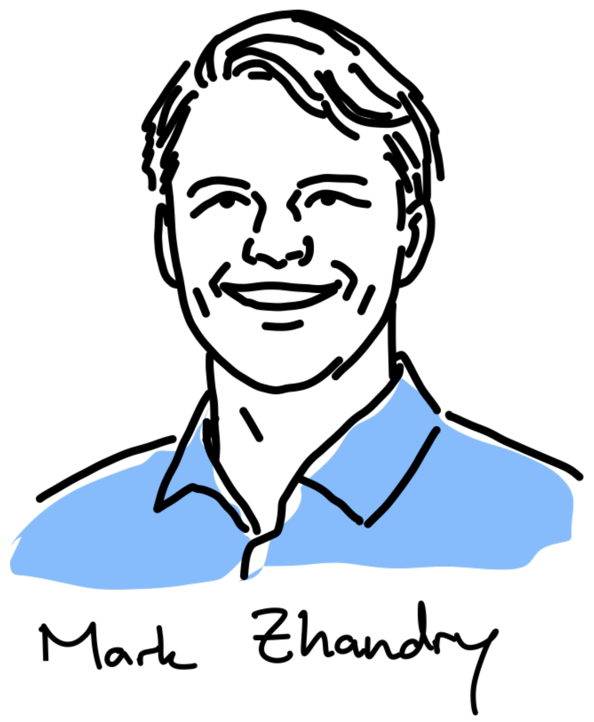
I.e, it also explains whether ground states can or cannot be efficiently described.

Evidence that QCMA $\neq$ QMA

Bostanci, Haferkamp, *Nirkhe,* Zhandry (STOC 2026).

- Proving QCMA $\neq$ QMA requires proving P $\neq$ PSPACE (open since 1960's)
- Instead, we prove that the classes are not equal in a query complexity model
- Many interpretations of this result
  - QCMA $\neq$ QMA for problems about *classical* black-box functions/oracles
  - QCMA $\neq$ QMA when Arthur's verification algorithm runs in logarithmic-time
  - QCMA cannot even approximate ground energy (QMA-complete) unless the algorithm strongly depends on the structure $\mathbf{H} = \sum_i h_i$.





The power of quantum witnesses

- What differentiates a quantum state $|\psi\rangle$ from a classical string w?
- **No-cloning theorem.** There is no general transform cloning quantum states:

$$|\psi\rangle \not\longrightarrow |\psi\rangle \otimes |\psi\rangle$$

- Whereas given a classical string w, we can make as many copies as we want!
- This is a way in which classical witnesses are more powerful than quantum ones!
- What we prove is that, paradoxically, this increase in power is too good to be true and therefore there're exist quantum problems without classical witnesses!

The power of quantum witnesses

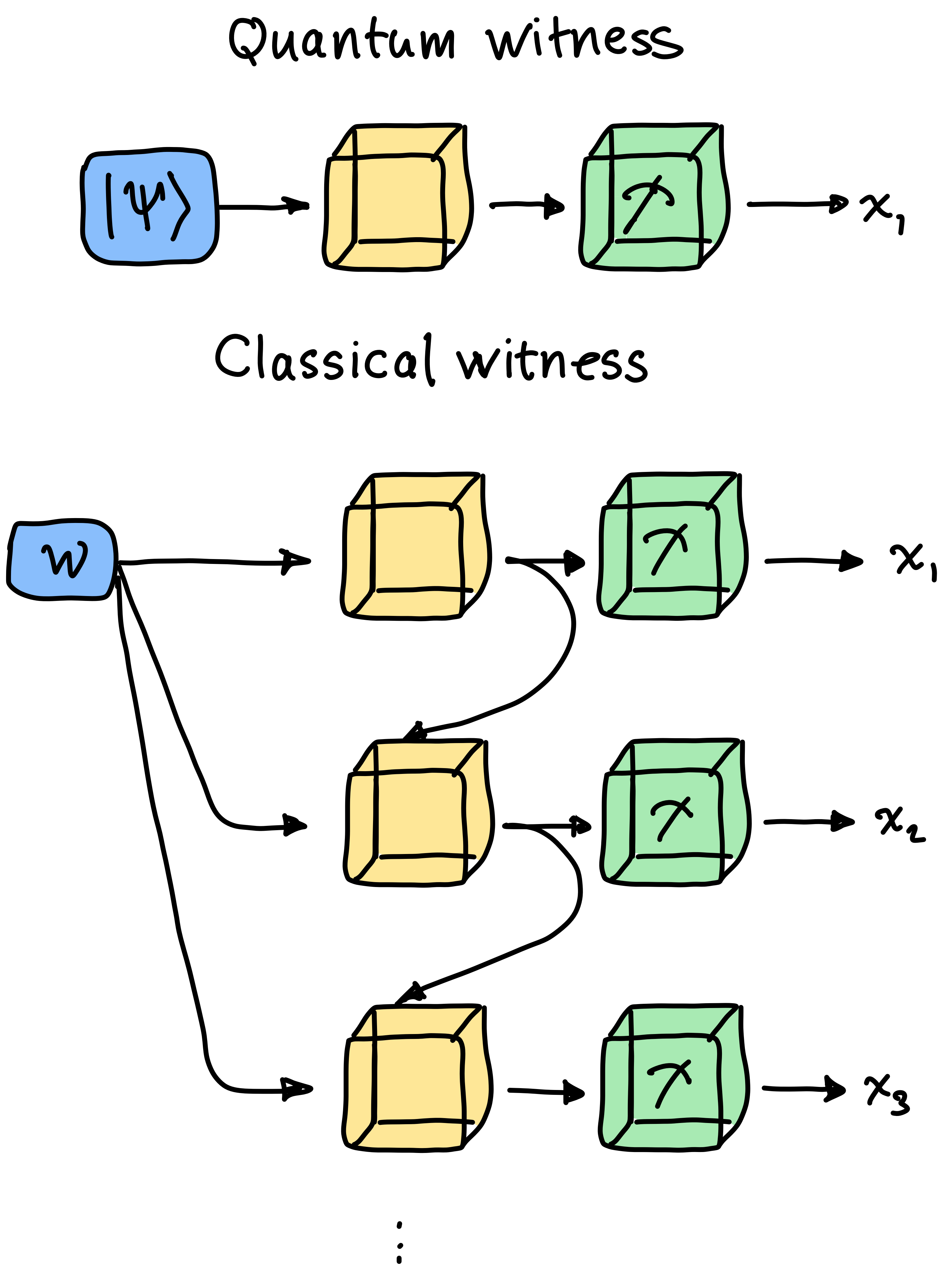
- What makes quantum and classical witness different?
- Quantum witnesses are use-once objects. Classical witnesses can be reused.

Given witness $|\psi\rangle$ and q. circuit V we can produce 1 sample from $p(x) = |\langle x|V|\psi\rangle|^2$

Given witness w and q. circuit V we can produce as many samples as we want from $p(x) = |\langle x|V|w\rangle|^2$

Because we can clone w and can't clone $|\psi\rangle$.

The power of quantum witnesses

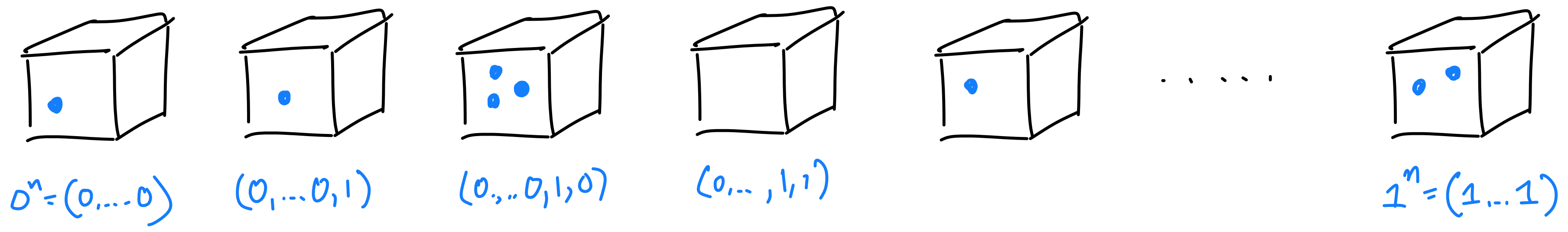


<u>Proof sketch of QMA v QCMA query separation</u>

Identify a distribution D that is provably hard to produce exp(n) many samples from

Argue that if QCMA = QMA (in query model) that we could construct such a sampler, a contradiction

Distribution is derived from the position vs momentum uncertainty principle for bosons

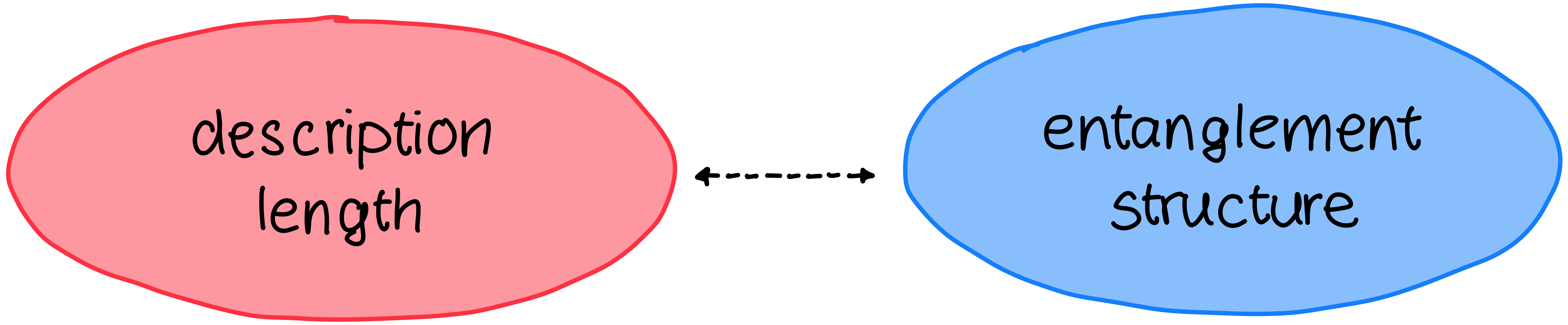


Part III: Robust entanglement

Ground energy approximations

- What have we seen so far?
  - Computing ground energy is QMA-complete due to Kitaev
  - And from $\mathsf{QMA} \neq \mathsf{QCMA}$ query proof, good evidence that ground states cannot be efficiently described classically
- **How hard is it to approximately compute the ground energy?**
  - How hard is it to describe approximations of the ground state?
  - Kitaev's lower bounds only extend to show hardness for describe approximations that are $O(1/n^3)$-close

The relationship between description complexity and entanglement



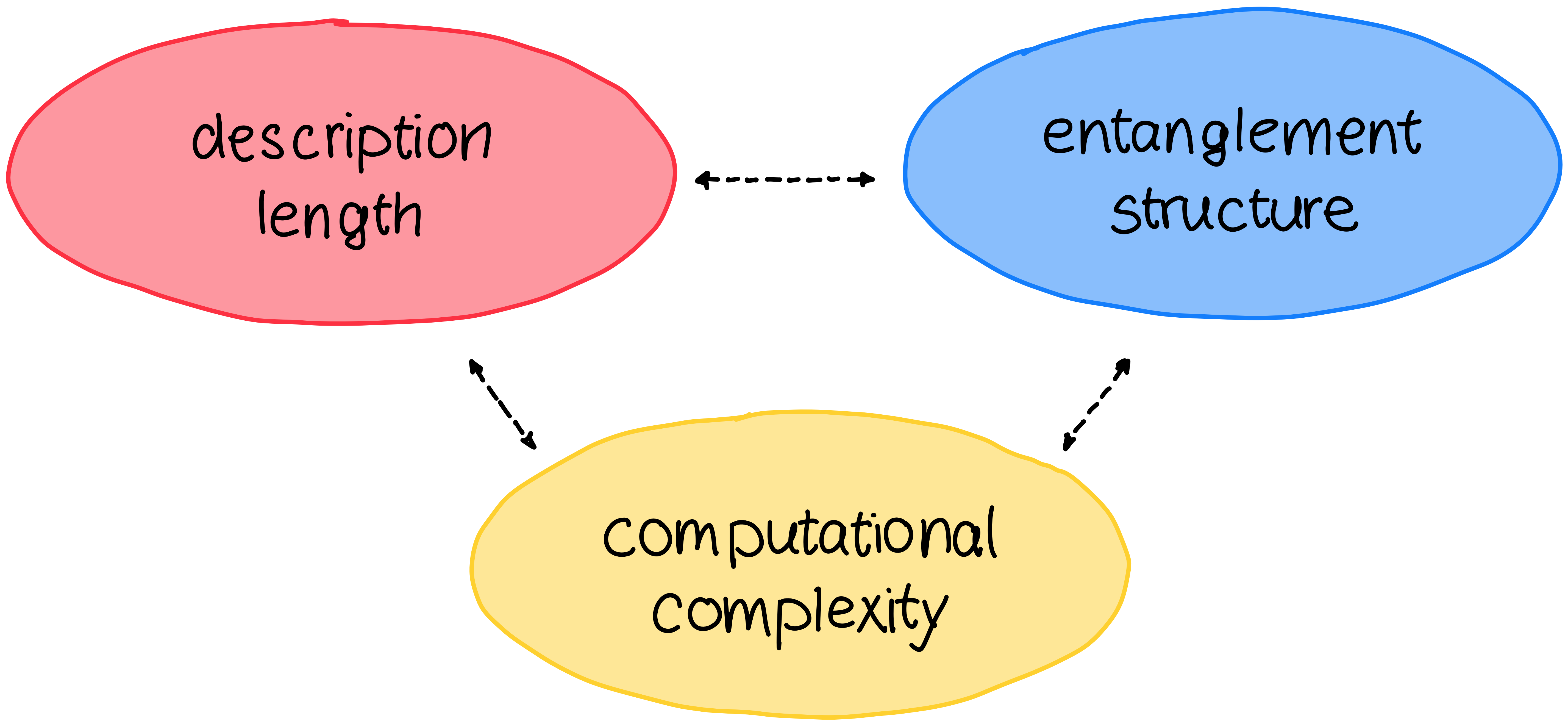
quantum entanglement

doesn't suffice to describe each individual particle

the longer the description $\Longrightarrow$ the more "complex" the entanglement

description length is a "measure" of entanglement complexity

The relationship between description complexity and entanglement

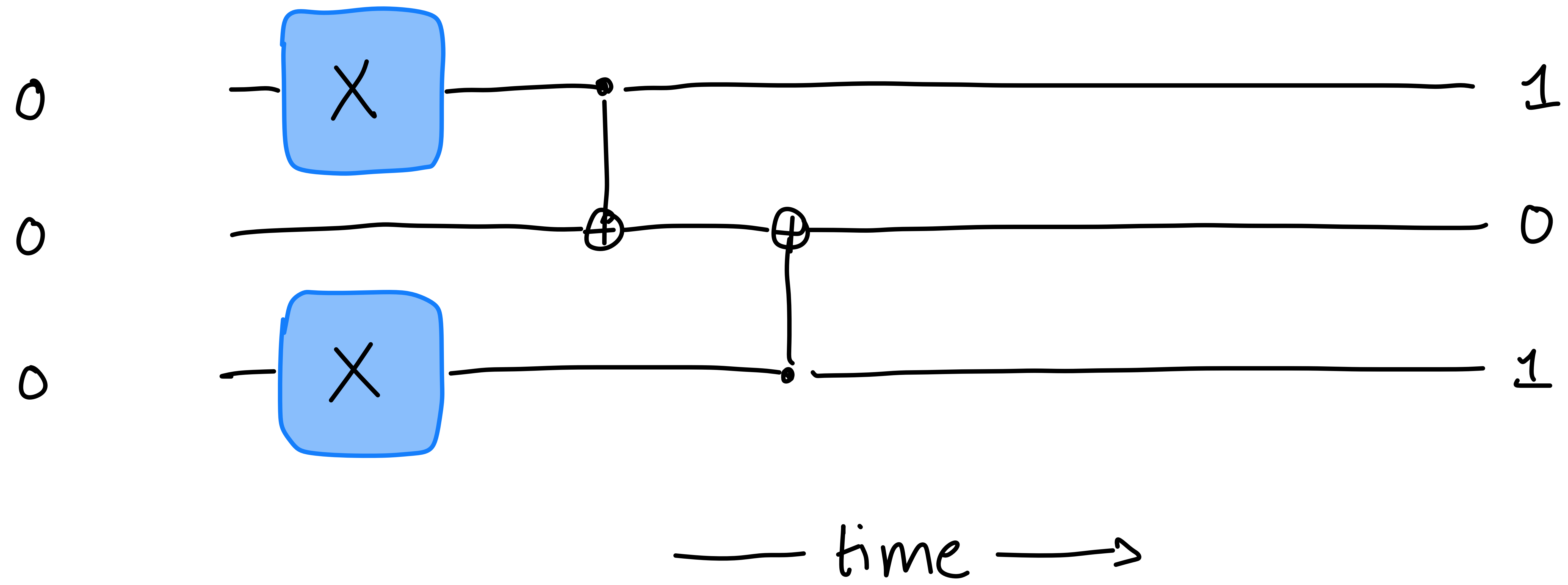


How do description length & computational complexity intersect?

The Circuit Model

The circuit model

Computation can be described in the *circuit* model



The circuit model

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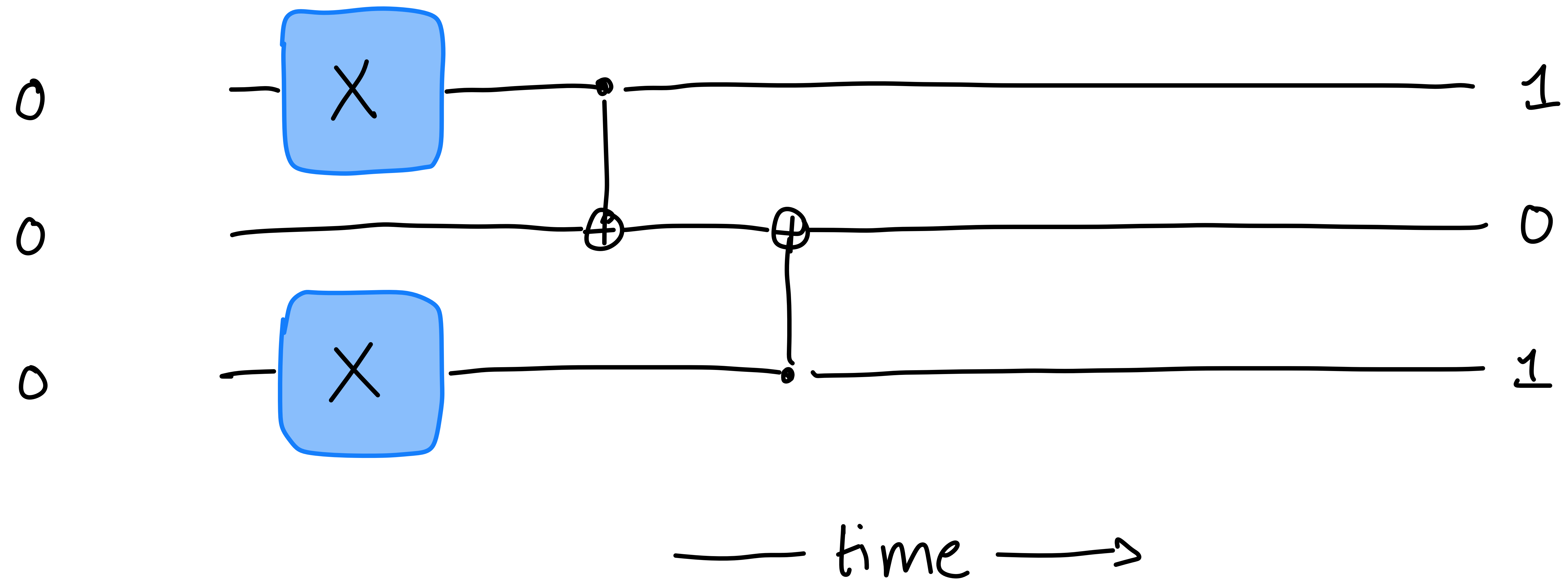
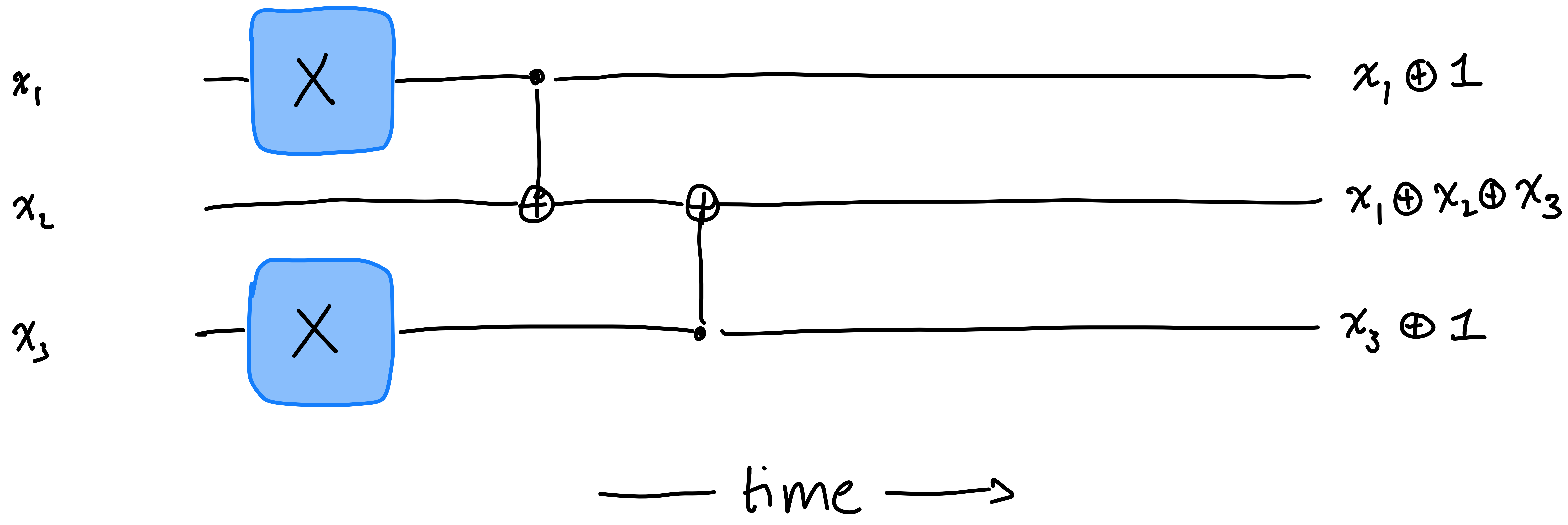


Computation can be viewed as an *input/output* function

The circuit model

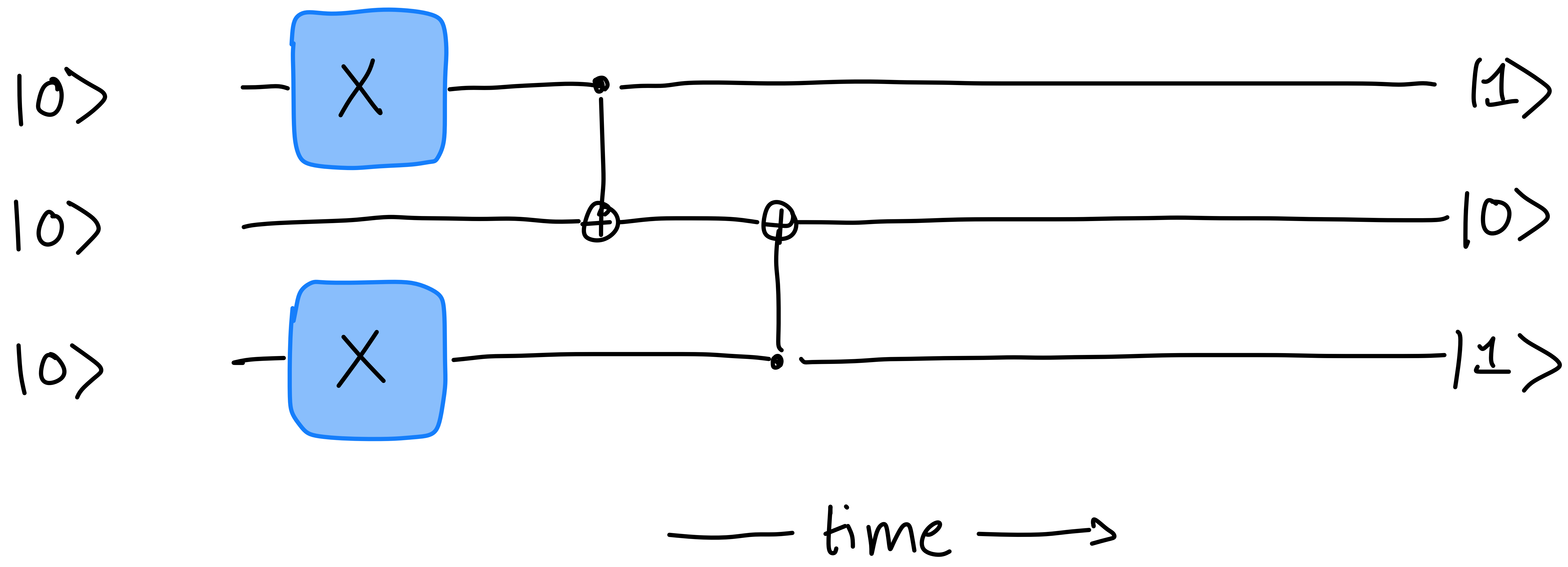
Computation can be described in the *circuit* model



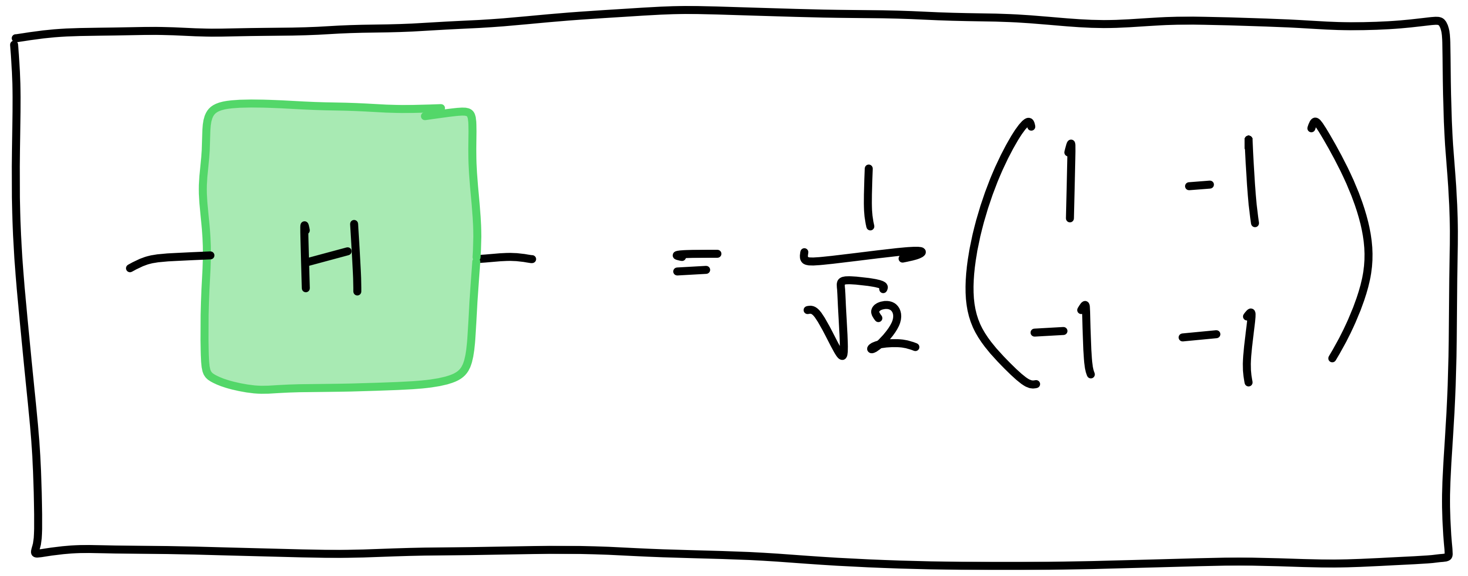
Computation can be viewed as an *input/output* function

The circuit model

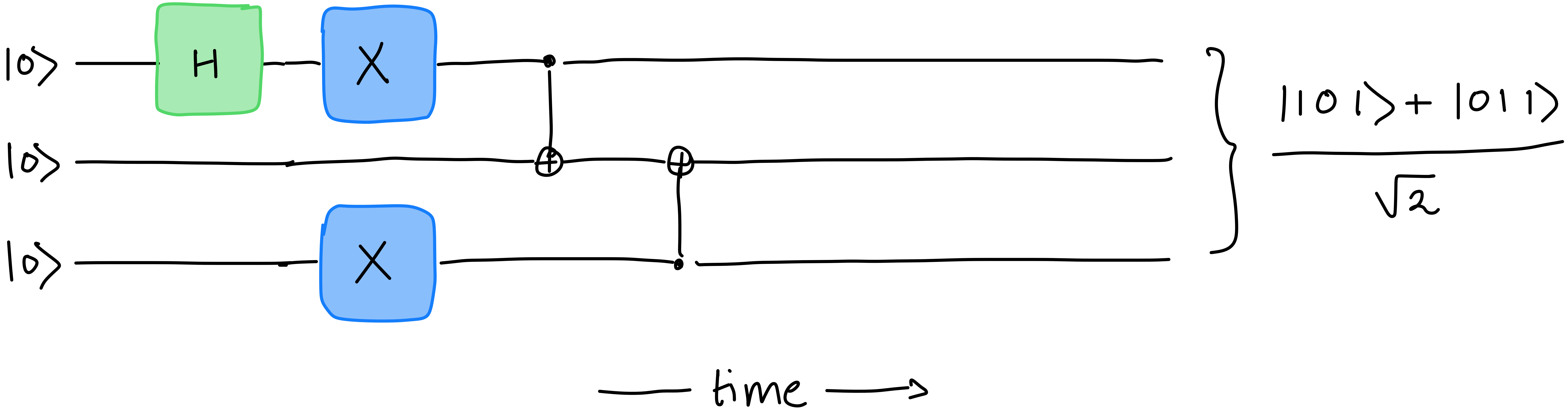
Likewise, quantum computation can be described in the *circuit* model



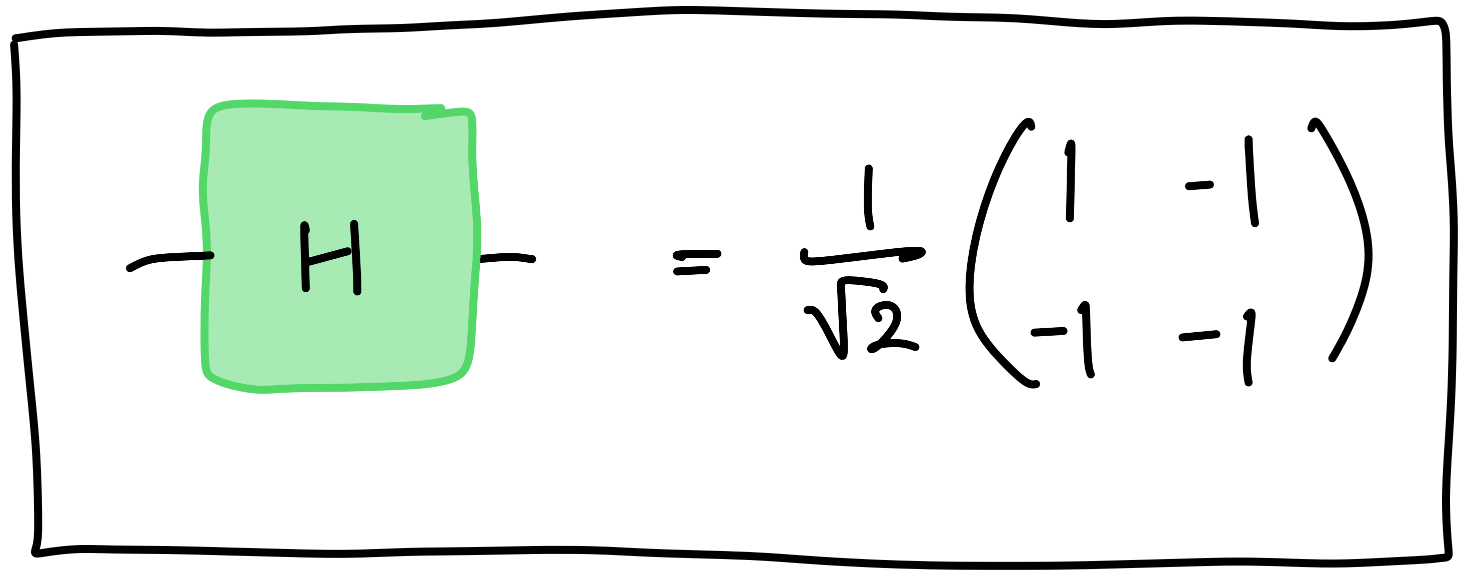
The circuit model



Likewise, quantum computation can be described in the *circuit* model



The circuit model



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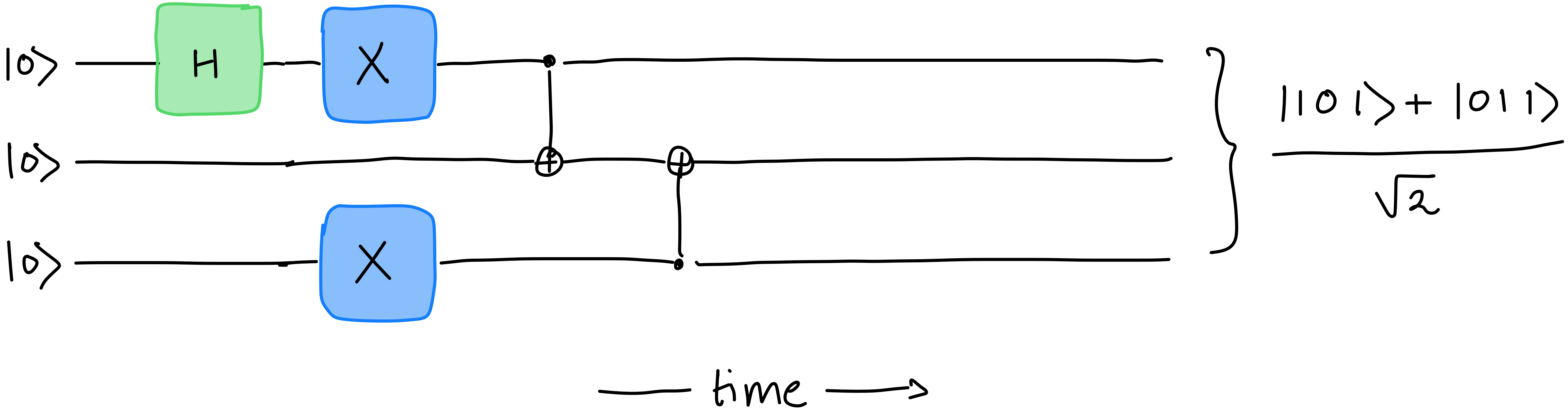
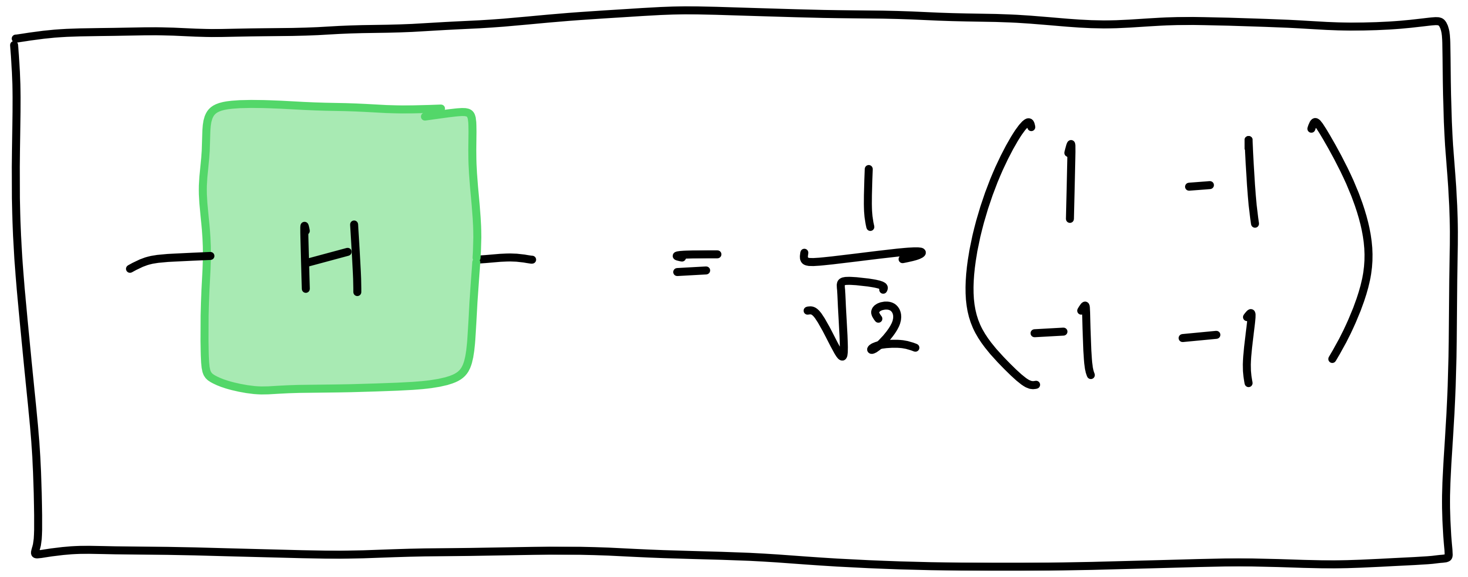


quantum computation can also be viewed as an *input/output* function

The circuit model



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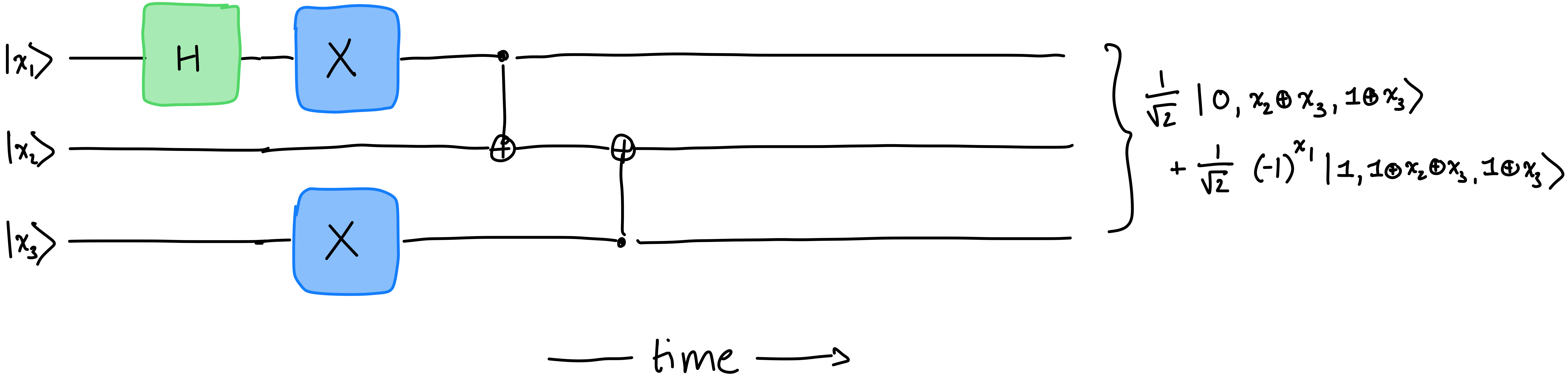

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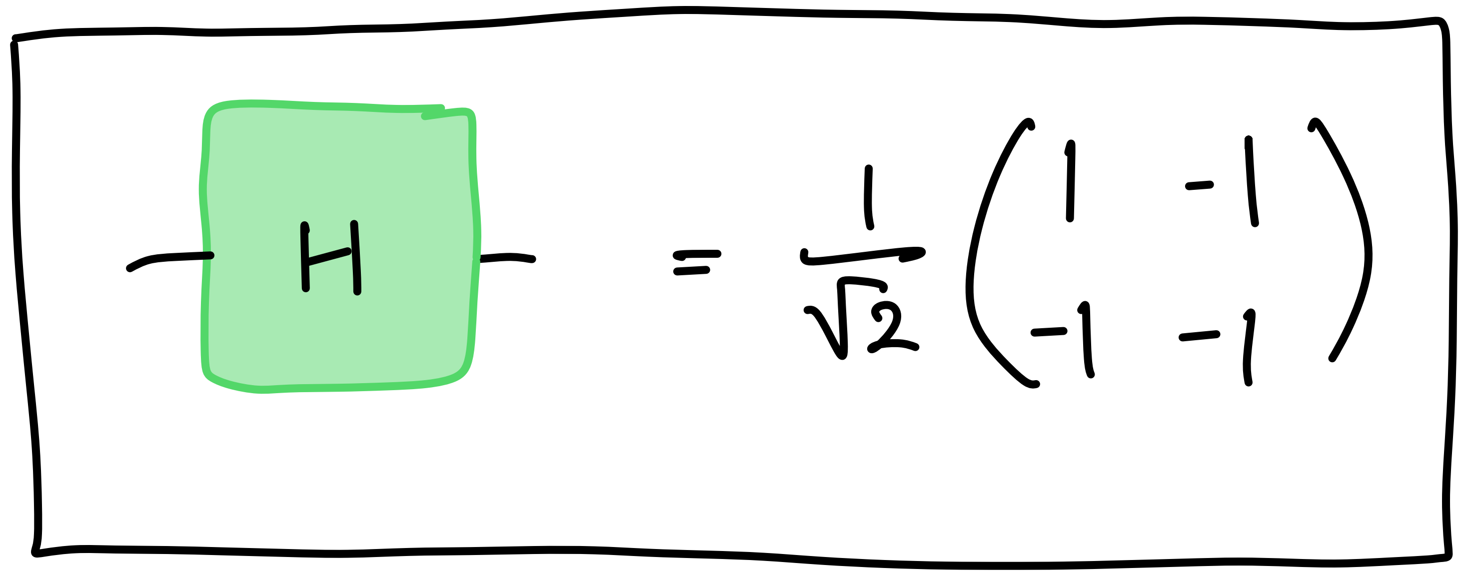

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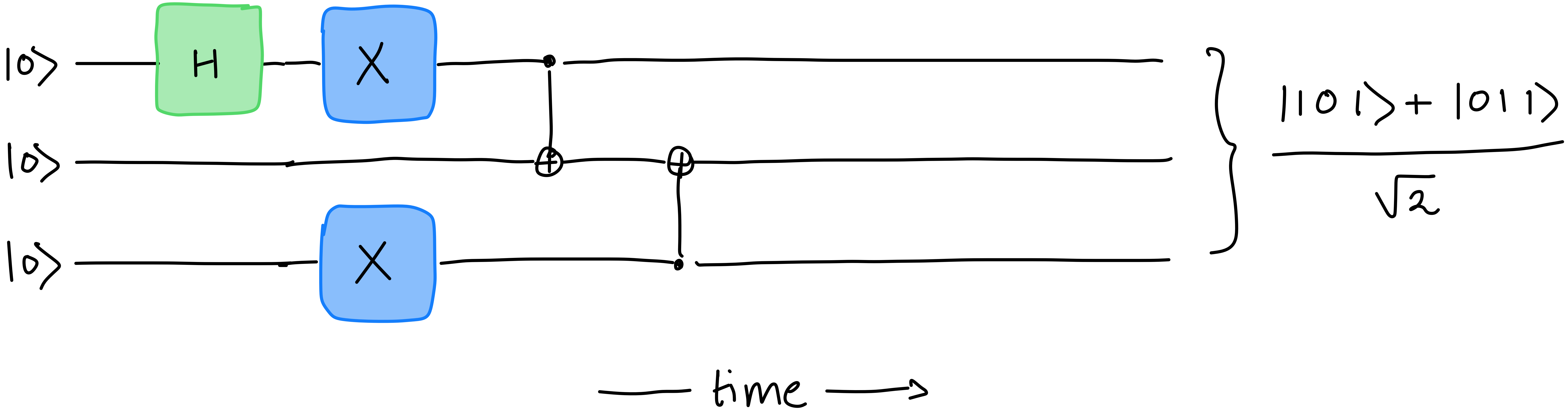
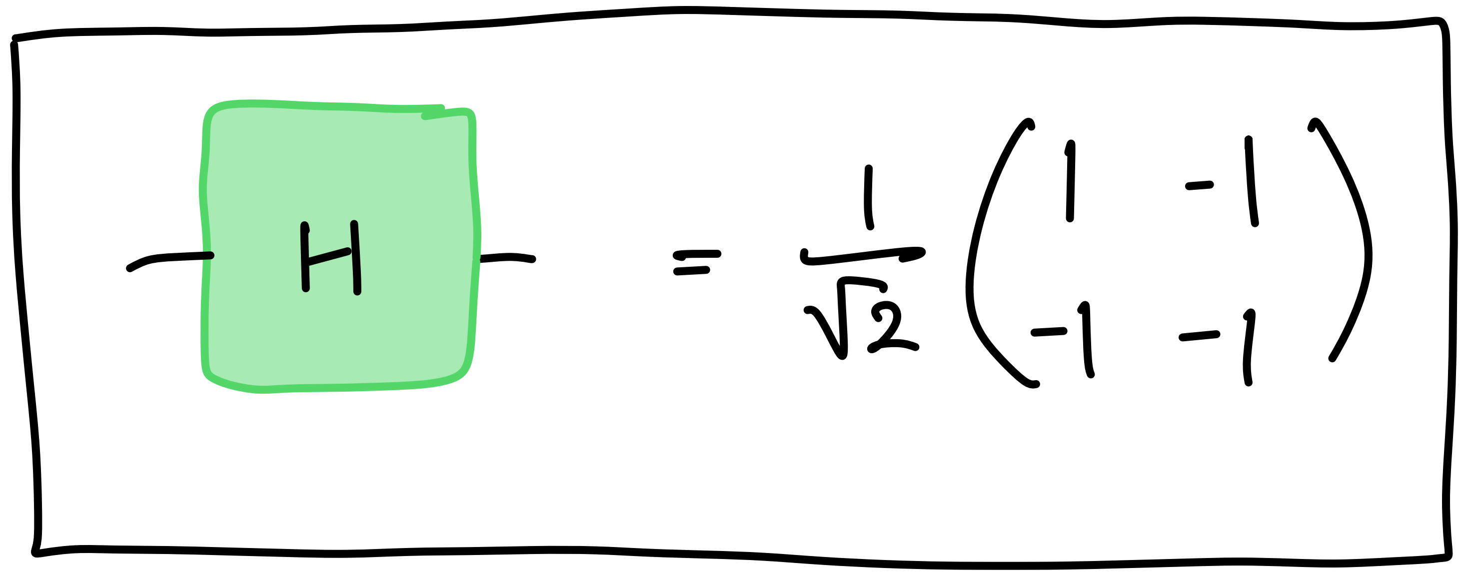
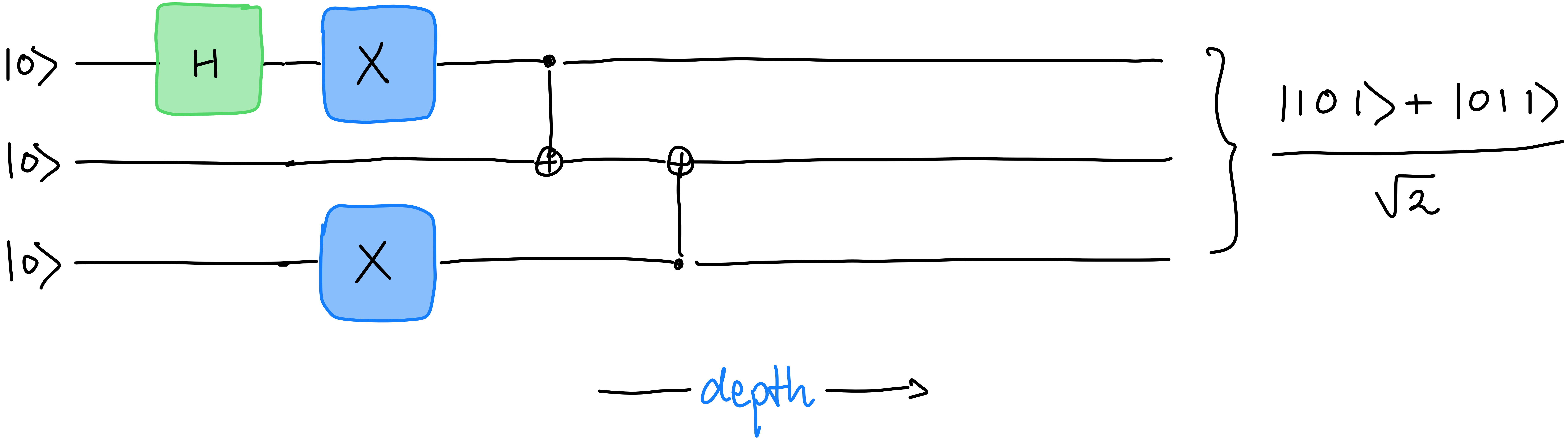


quantum computation also gives us a measure of complexity for states

The circuit model



Likewise, quantum computation can be described in the *circuit* model

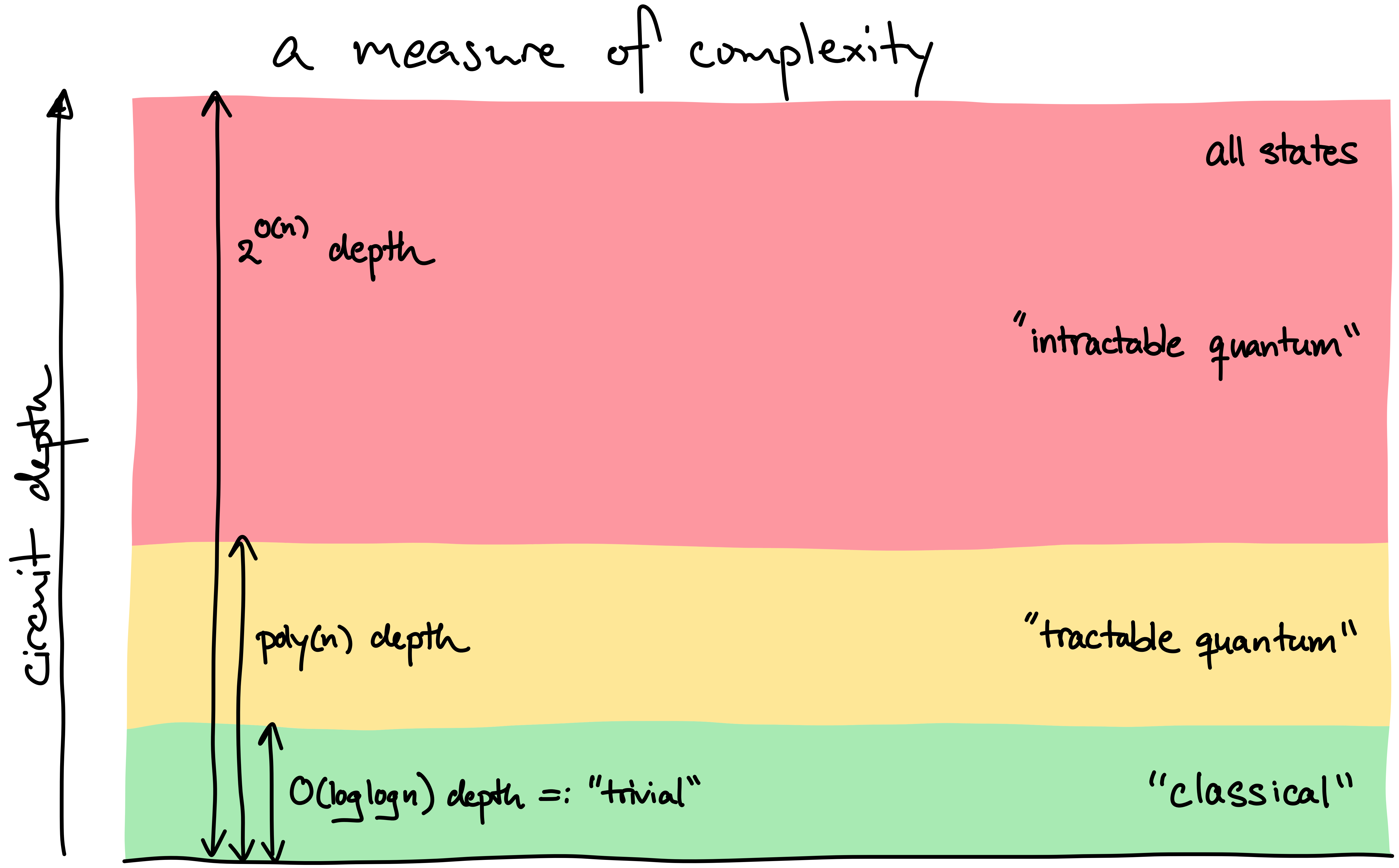


quantum computation also gives us a measure of complexity for states

depth($|\psi\rangle$) = min depth over all circuits generating $|\psi\rangle$ from $|0^n\rangle$

The quantum (computational) yardstick

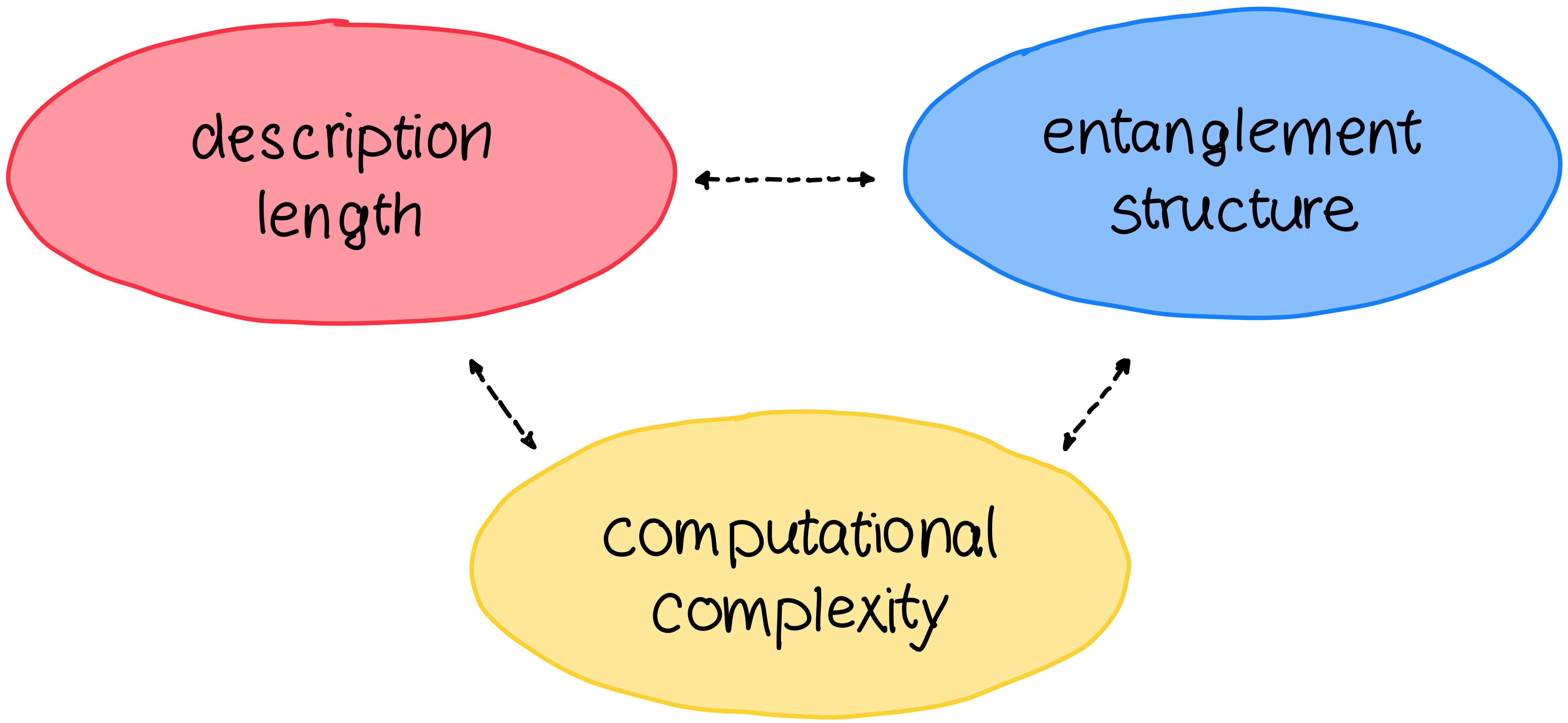
depth($|\psi\rangle$) = minimum depth of circuit with output $|\psi\rangle$



efficient quantum algorithm for calculating the energy of $|\psi\rangle$ for any Local Hamiltonian H

efficient classical algorithm for calculating the energy of $|\psi\rangle$ for any local Hamiltonian H

The relationship between description complexity and entanglement

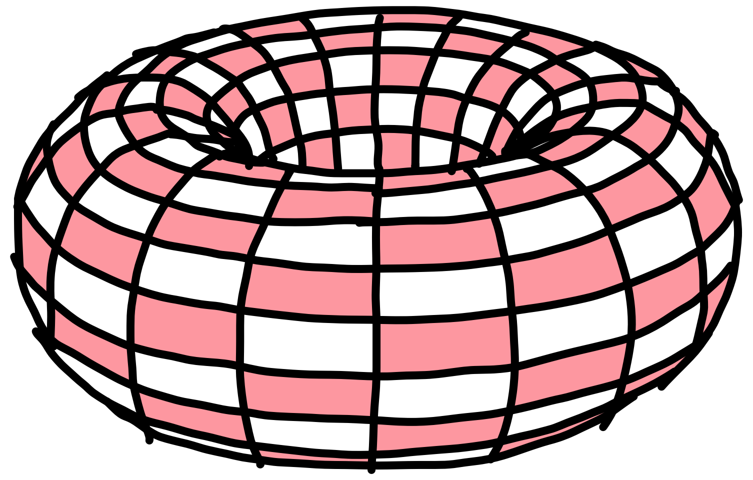


Do all local Hamiltonian systems have low depth circuit approximations for the ground state?

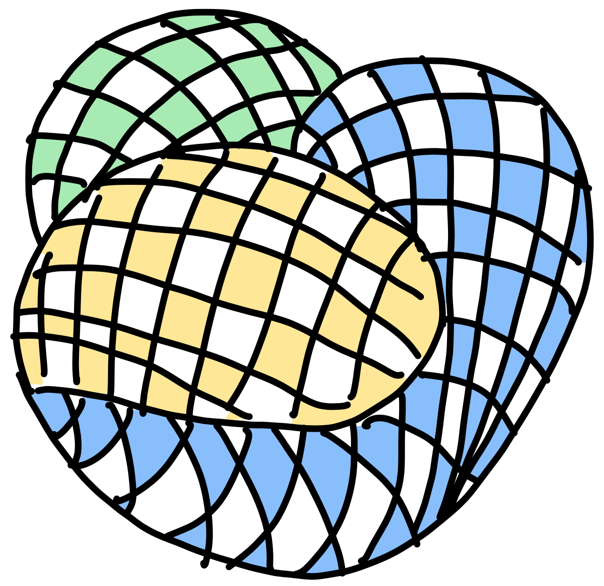
Inapproximable local Hamiltonian systems

NLTS theorem: Anshu, Breuckmann, *Nirkhe* (STOC 2023).

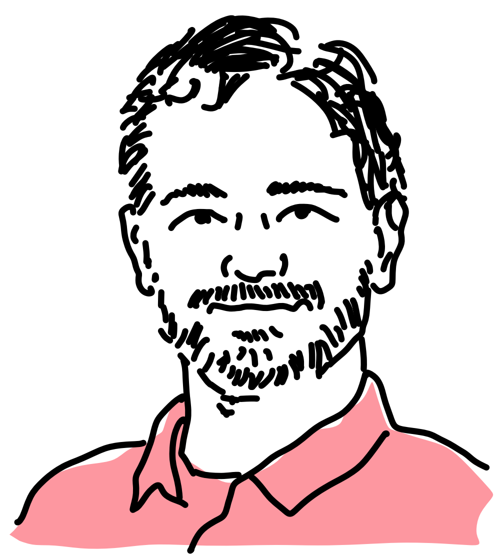
- Most physical systems (ex. Toric code, 2D ising models, AKLT models, etc.) have approximations that can be described by low-depth circuits
- **Theorem:** But we can *engineer* systems for which all the $\epsilon -$ approximations of the ground state require circuit depth $\Omega(\log n)$ for n-qubit Hamiltonians
- Robust entanglement can (potentially) exist at “warm” temperatures!
- This is an *engineering* feat. —construction built from mathematical tools like error-correction and expander graphs
- We believe stronger lower bounds are possible —some evidence of this in restricted models of computation



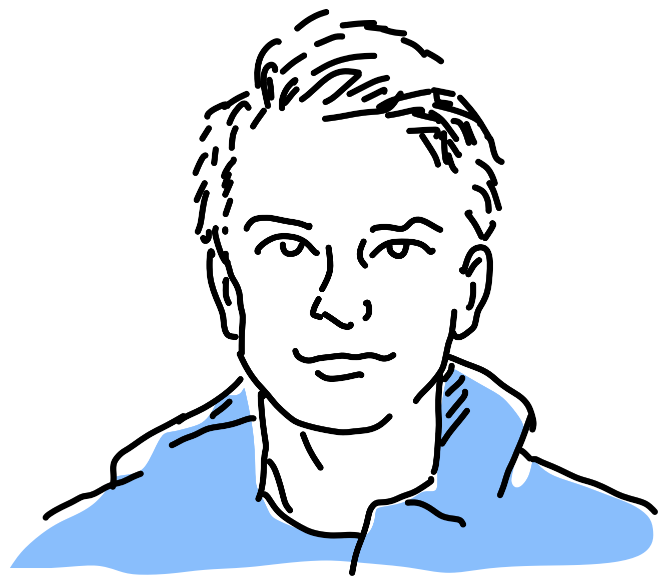
Hamiltonian on a physical system on a torus



The Hamiltonian we engineer lives on a high-dimensional manifold with an expander-like topology. This is a cartoon of it folding in on itself.



Anurag Anshu



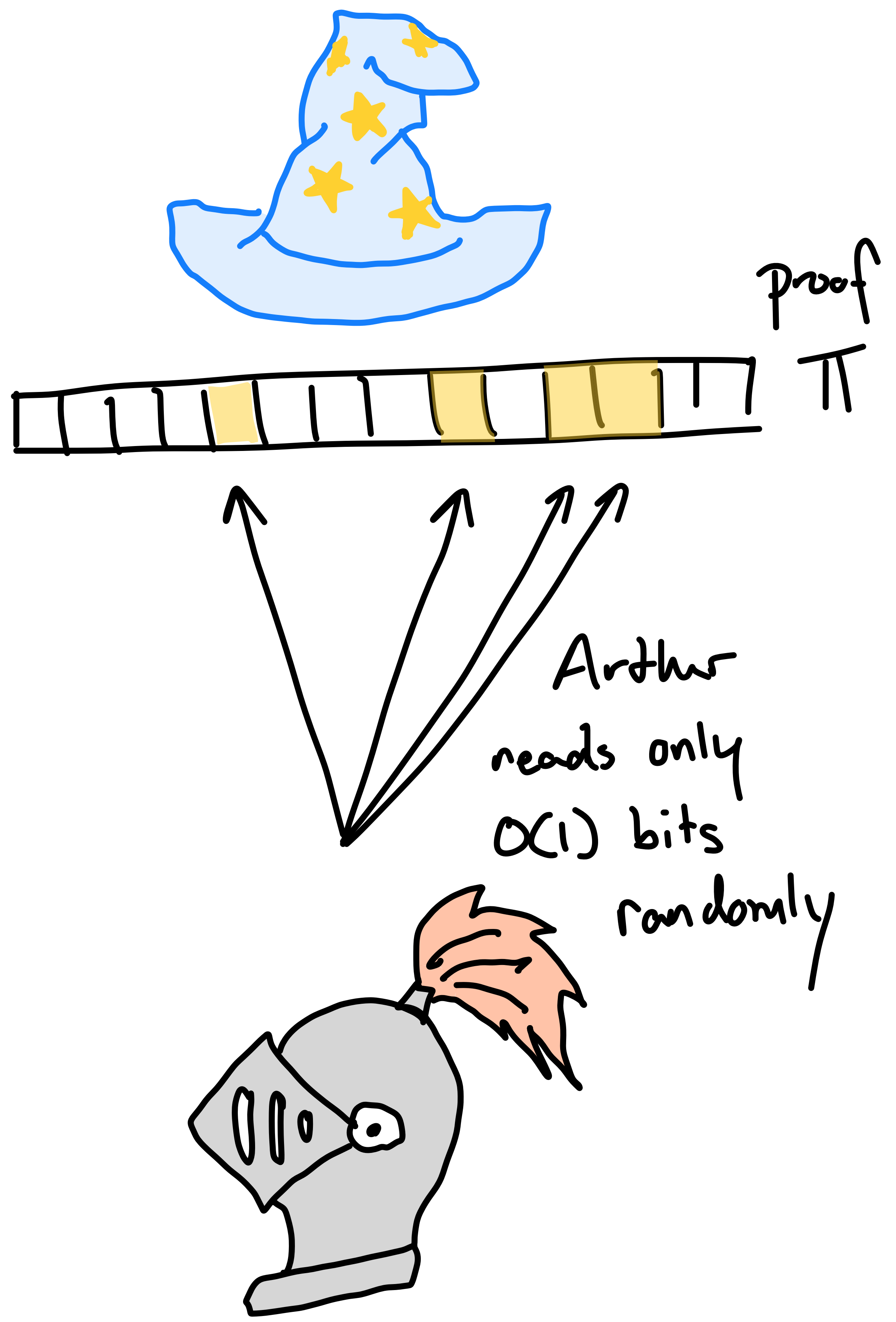
Niko Breuckmann

Chinmay Nirkhe

Part IV: The open questions

The Quantum PCP conjecture

- In 1992, Arora-Lund-Motwani-Sudan-Szegedy, had a breakthrough in classical computer science.
  - They were studying the complexity of approximating NP problems like 3-SAT due to breakthroughs in a field called *interactive proofs*
  - They proved that approximating the number of 3-SAT equations satisfiable to < 12.5% is as hard as solving the problem exactly
    - In other words, the approximation problem is NP-complete
- This was a major breakthrough — proving why no good approximation algorithms had been found for these fundamental problems
- This result is called the **probabilistically checkable proofs theorem** due to an alternate interpretation as Arthur verifying a classical proof sent by Merlin approximately



The Quantum PCP conjecture

- The question of whether a quantum analog of the PCP theorem is the most outstanding open question in quantum complexity theory
- First posed by Aharanov in 2002, many alternate formulations, consequences, variants have been studied (including many by myself)

QPCP Conjecture (satisfiability) [AN'02]: **QMA**-hard to estimate the ground-energy of a local Hamiltonian up to 1% multiplicative error

+ $\mathrm{NP} \neq \mathrm{QMA}$ ⇒

NLTS Theorem [ABN23]: There exist local Hamiltonian systems with every $\leq 0.01n$-energy state requiring at least $\Omega(\log n)$ circuit depth

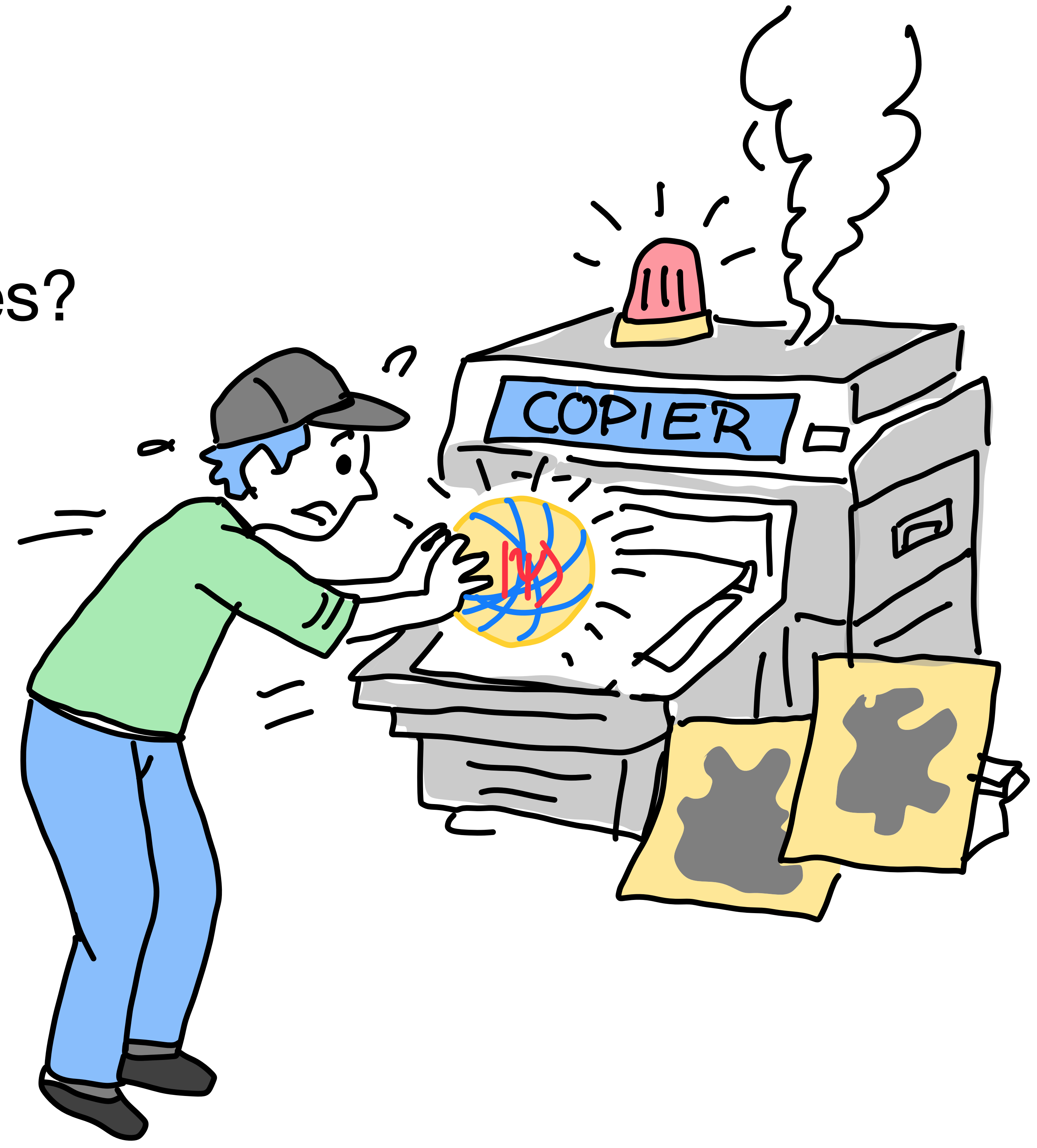
QPCP Conjecture (satisfiability) [AN'02]: **QMA**-hard to estimate the ground-energy of a local Hamiltonian up to 1% multiplicative error

+ $\mathrm{QCMA} \neq \mathrm{QMA}$ ⇒

Conjectured lower-bound: There exist local Hamiltonian systems with every $\leq 0.01n$-energy state requiring at least $\omega(\mathrm{poly}(n))$ circuit depth

Complexity theory → Cryptography & Quantum Engineering

- Complexity theory gives us evidence of complex entanglement
- Can we engineer that complex entanglement to *engineer* interesting states?
- **Quantum cryptography**
  - Use entanglement to create new cryptographic primitives
  - Key desired properties: unforgability, unclonability, use-one, etc.
- **Quantum engineering**
  - Design robust entanglement for better quantum computations, reduce communication overheard, fault tolerance
  - Understand the nature of entanglement by explore what can be engineered



Last thing … a shameless plug for UW

- If you are interested in researching more about quantum in general … be it from a computational, physics, chemistry angle, the University of Washington is a great place to consider for graduate school!
- Multiple programs and a large investment by the region in quantum computation/technologies
- cs.washington.edu
  theory.cs.washington.edu | quantum.cs.washington.edu

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