Classical descriptions of quantum states

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Based on work with

Irani, Natarajan, Rao & Yuen

&

Arunachalam, Bravyi, & O'Gorman

How does one <u>describe</u> a quantum state?

How does one <u>use</u> a description of a quantum state?

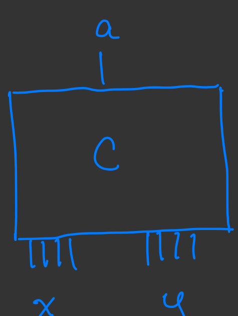
Do quantum problems of classical description length ℓ have classical solutions of length poly(ℓ)? (QCMA vs QMA)

- If not, what is the shortest length of a solution to the problem? What about complexity notions?

A motivation for complexity of sols. vs problems.

Thm (Impagliazzo - Wigderson) unless $NEXP \subseteq \Sigma_2 \subseteq PH$, Succinct-3-coloring (NEXP-complete) does not have succinct solutions!

Succinct-3-coloring: Input: $\langle C \rangle \leftarrow$ circuit description



$\in \{0,1\}^n$

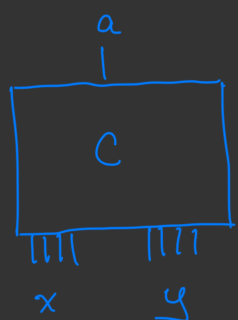
G = graph implicitly defined by C.

edge $x \sim y \iff C(x,y) = 1$.

Goal: Decide if G is 3-colorable.

A motivation for complexity of sols. vs problems.

Succinct-3-coloring: Input: $\langle C \rangle$ ← circuit description



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G = graph implicitly defined by C.

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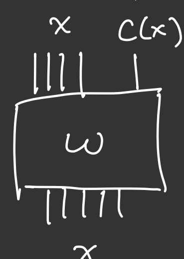
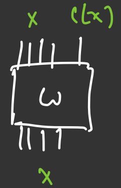
Goal: Decide if G is 3-colorable.

Say S3COL instance $\langle C \rangle$ has a succinct sol. if $\exists$ poly sized ckt

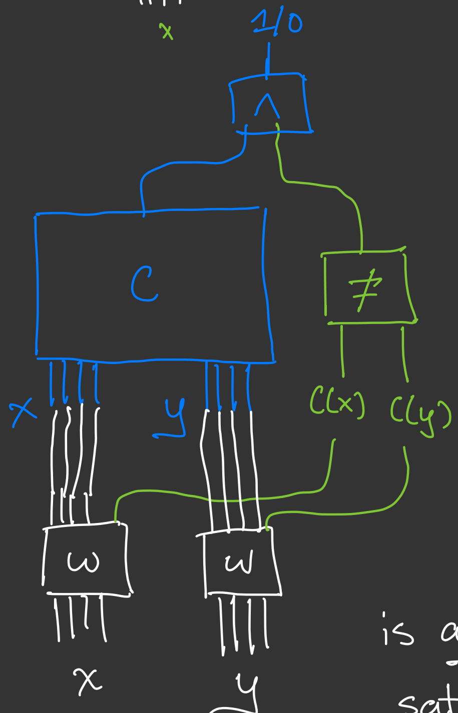


outputting coloring $c(x)$ from optimal coloring.

Thm S3COL doesn't have succinct sols.

Pf If $\exists$ 

then

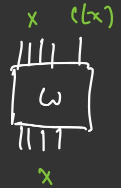


is always satisfiable.

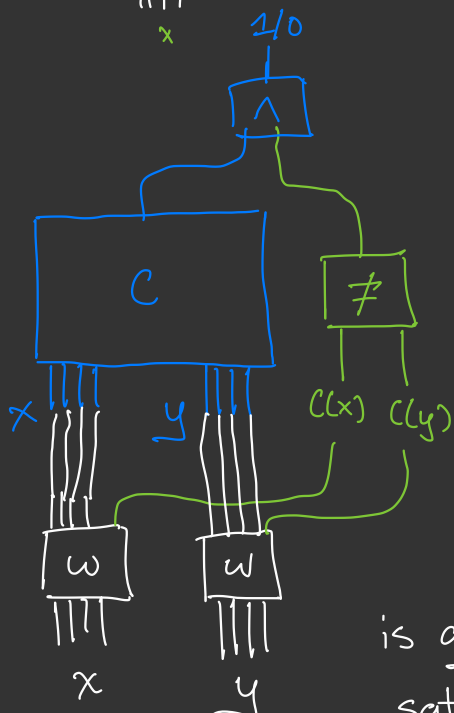
output 1 if

$x \sim y$ AND $c(x) \neq c(y)$.

<u>Thm</u> S3COL doesn't have succinct sols.

<u>Pf</u> If $\exists$ 

then



is <u>always</u> satisfiable.

Then $\langle C \rangle \in$ S3COL iff

$\exists \langle w \rangle$ s.t. BIG-CKT is <u>always</u> satisfiable.

i.e. $\exists\, w$ s.t. $\forall\, x, y$

$$B(x, y, w) = 1$$

$\Rightarrow$ NEXP $\subseteq \Sigma_2 \subseteq$ PH.

Why is this classical CS textbook pf important?

It provides a clear separation between the description complexity of sols. and questions.

Notice, that even with a succinct description of S3COL we would not expect to check the problem in sub-exponential time.

exponential time classes

$P \subseteq NP \subseteq \Sigma_2 \subseteq PH \subseteq \ldots \subseteq NEXP$

Instead, description complexity yields a speedup among these large complexity classes that all take exponential time.

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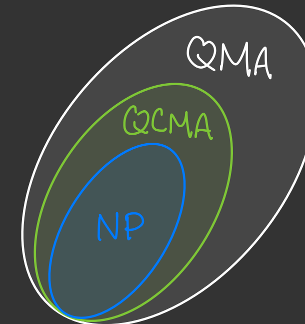
Today's talk: How should we define description complexity for quantum problems and what is known?

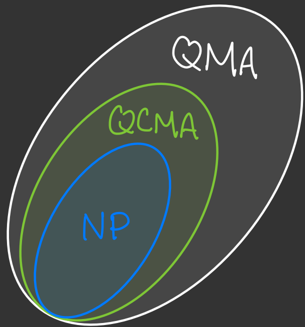
Non-deterministic quantum computation



QMA $\overset{?}{=}$ QCMA : Do all "classically describeable" quantum questions have "classically describable" solutions?

Note: both cases still speculate the problem is exp-hard for BPP (or BQP). It's a matter of description.

Non-deterministic quantum computation



$QMA \overset{?}{=} QCMA$: Do all "classically describeable" quantum questions have "classically describable" solutions?

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If $QCMA \neq QMA$, what complexity class captures the classical complexity of solutions to QMA problems?

Search-to-decisions:

How much harder is finding a solution than deciding if one exists?

For the class NP, it's equally hard...

$$\exists x_2 \dots x_n, \ \varphi(0, x_2, \dots, x_n) = 1$$

yes
set $y_1 = 0$

no
set $y_1 = 1$

$$\exists x_3 \dots x_n, \ \varphi(y_1, 0, x_3, \dots, x_n) = 1$$

yes
set $y_2 = 0$

no
set $y_2 = 1$

$\vdots$

End of process,
$y_1 \dots y_n$ forms a sol.
to φ.

What about quantum search-to-decision?

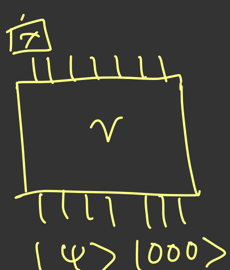
First What does search-to-decision mean in this context?

Issues: 1. QMA is a promise class.

2. The solution might depend on the verifier.

Search QMA def:

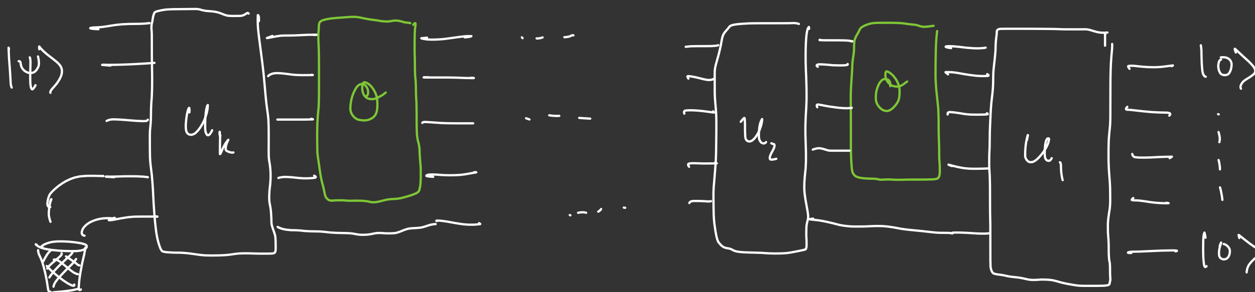
Given a canonical QMA problem described as a verifier V



Output a state $|\psi\rangle$ which that verifier will accept with prob. $\frac{2}{3}$.

QMA search-to-decision reductions

Input: Verifier ckt V.



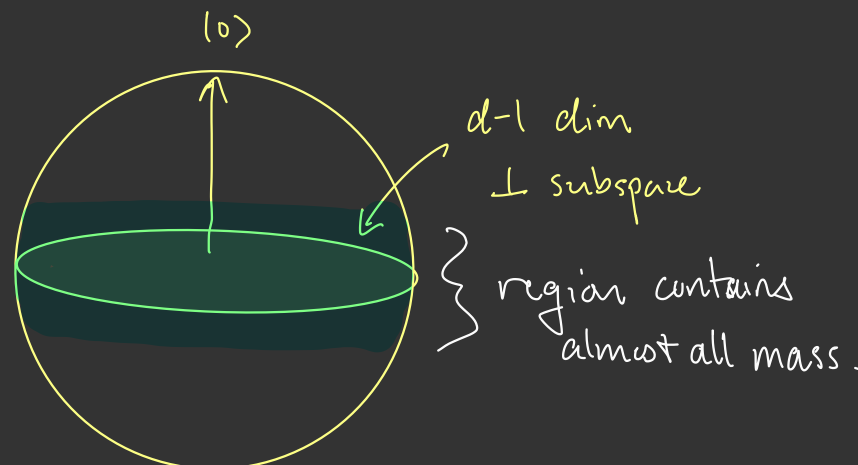
Circuit with oracle gates O accessed in superposition

$$O(x) = \begin{cases} 1 & \text{if } x \text{ encodes a YES QMA question} \\ 0 & \text{if } x \text{ encodes a NO QMA question} \\ \text{either} & \text{if } x \text{ encodes an } \underline{\text{invalid}} \text{ QMA question} \end{cases}$$

Goal: Output $|\psi\rangle$ accepted w pr $\frac{2}{3}$ by V.

Difficulties to overcome

There is no good way to binary search over the Hilbert space.



Trying to find $|\psi\rangle$ by a seq. of projectors is a no-go path.

"entanglement destroying"

(also why ground-space dim counting seems hard).

Thm (Aaronson / Folklore)

$\exists$ a $2n+1$ query algorithm for generating any state $|\psi\rangle$ up to $\exp(-n)$ accuracy.

(When applied to QMA sols., oracle complexity = PP.)

Our theorem Thm (INN+'21)

$\exists$ a 1-query PP algorithm for generating the sol. to QMA problems.

(We also have extensions to general states).

Crucial intuitions

① Building all states is unnecessarily powerful.

By counting, there are only $2^{\mathrm{poly}(n)}$ QMA problems $\ll$ $\exp(-n)$ net over $\mathcal{H}$.

② Since QMA states are verifiable, ~~signs~~ real vs. imaginary-ness of amplitudes don't matter.

$|\psi\rangle = \sum_x \alpha_x |x\rangle$, then $\exists$ $|\phi\rangle = \sum_x \beta_x |x\rangle$ $\beta_x \in \mathbb{R}$

s.t. $|\langle\phi|\psi\rangle| \geq$ constant.

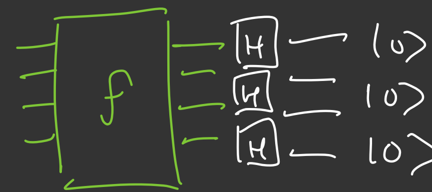
③ If $|\psi\rangle$ is Haar-random, then the amplitudes concentrate around $\frac{1}{\sqrt{2^n}}$.

$$\mathop{\mathbb{E}}_{|\psi\rangle \sim \mathrm{Haar}} |\langle x|\psi\rangle|^2 = \frac{1}{2^n}, \quad \mathop{\mathbb{E}}_{|\psi\rangle \sim \mathrm{Haar}} |\langle x|\psi\rangle|^4 = \frac{2}{2^n(2^n+1)}.$$

Interlude: Phase states.

$f : \{0,1\}^n \to \{0,1\}$

$$|\psi_f\rangle = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle =$$



For any vector $|v\rangle \in \mathbb{R}^{2^n}$, best phase state approx $|v\rangle$ is with $f(x) = \mathrm{sgn}(\langle x|v\rangle)$.

$$\Rightarrow \quad \langle \psi_f | v \rangle = \frac{\| |v\rangle \|_1}{\sqrt{2^n}}.$$

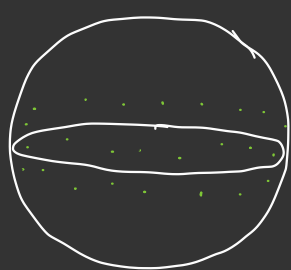
Lem $\quad \Pr_{|v\rangle \sim \mathrm{Haar}} \left[\frac{\| |v\rangle \|_1}{\sqrt{2^n}} < \frac{\sqrt{\alpha}}{2} \right] < \alpha.$

Phase states (cont.)

Lem
$$\Pr_{|v\rangle \sim \text{Haar}}\left[\frac{\| |v\rangle \|_1}{\sqrt{2^n}} < \frac{\sqrt{\alpha}}{2}\right] < \alpha.$$

$$\Pr_{f,g}\left[|\langle \psi_f | \psi_g \rangle| > \delta\right] \le 2\exp\left(-\frac{\delta^2 \cdot 2^n}{3}\right)$$
Chernoff bound.

In short, phase states form an effective net for the Hilbert space under the Haar measure.



goal: show PP fn f s.t. $|\psi_f\rangle$ approximates QMA sol.

Small issues to handle

① Sol. $|\tau\rangle$ may not be approximable by phase states.

But for Clifford C, $C^\dagger H C$ will be whp.
Then can rotate phase state by $C^\dagger$ to recover.

② To define fn $f(x) = \mathrm{sgn}(\mathbb{R}(\langle x|\tau\rangle))$ we need $|\tau\rangle$. But,

$$|\tau\rangle \propto (\mathbb{1}-H)^{\mathrm{poly}(n)} \underbrace{D|0^n\rangle}_{\text{random clifford state}}$$

$f(x) =$
$\mathrm{sgn}(\mathbb{R}(\langle x|C^\dagger(\mathbb{1}-H)^p D|0^n\rangle))$.

<u>Thm</u> 1 query PP alg which outputs a state $|\psi\rangle$

s.t. $|\langle\psi|\tau\rangle|^2 \geq 2^{-10}$ whp.

- Can add phase estimation to either output $|\tau\rangle \pm \frac{1}{\text{poly}(n)}$ w pr 2^{-10}.

- Algorithm is parallelizable with still one query to boost success prob. to $1 - \frac{1}{\text{poly}(n)}$.

Is this the best we can do?

Oracle no-go result for QMA-search to QMA-decision reduction.

Thm (INN+'21)

$QMA^{\mathcal{O}}$ search problem with no $QMA^{\mathcal{O}}$ decision oracle alg.

$\mathcal{O} = \mathbb{1} - 2|\psi_f\rangle\langle\psi_f|$ where $|\psi_f\rangle$ is a phase state

OR $\mathcal{O} = \mathbb{1}$. Problem: Decide which scenario.

Idea All sols. accepted w pr $\geq \frac{2}{3}$, have large support on $|\psi_f\rangle$.

Oracle no-go (cont.)

Pf sketch: (1) Assume $\exists$ alg $\mathcal{A}^{QMA^{\mathcal{O}}, \mathcal{O}}$ that produces $|\psi_f\rangle$.

(2) Show that when run on $\mathcal{O}' = \mathbb{1} - |\psi_g\rangle\langle\psi_g|$, alg's step-by-step behavior is similar (hybrid alg).

(3) Argue whp should output nearly $\perp$ states and yet cannot by hybrid alg.

Consequences

QMA sols. can be described by phase states corresponding to PP fns ($2^{\mathrm{poly}(n)}$ PP fns and $2^{\mathrm{poly}(n)}$ QMA problems)

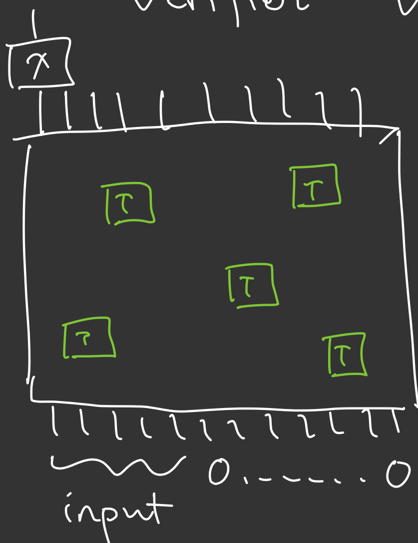
vs. 2^{2^n} phase states in general

$\nexists$ any hope for search-to-decision reductions for generic phase states (which are sols. to QMA^{O} problems)

Due to similarity of oracle separating QCMA/QMA, we suspect same oracles show S-2-D No-go's.

And now for a different angle on the same problem

What can we say about the complexity of sols. when verifier V has t T-gates?



Well known that deterministic q.c. with t T-gates requires $2^{O(t)} \cdot \mathrm{poly}(n)$ time.

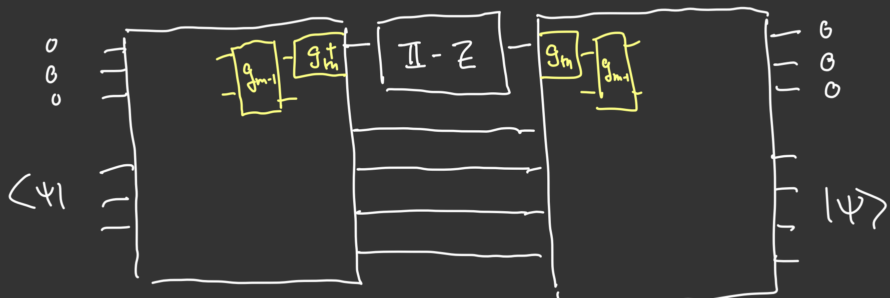
Thm (ABNO'21)

The optimal state $|\tau\rangle$ maximizing $|\langle 0|V|\tau\rangle|^2$ has form $W|\psi\rangle$ where $|\psi\rangle \in (\mathbb{C}^2)^{\otimes t}$ and W is a Clifford computable by prover and verifier.

A paramitrized approach to QMA

Alternate perspective: A reduction from QCSAT to Pauli Hamiltonian problem on t qubits with $\leq 2^t$ terms.

Pf sketch If V has 0 T-gates and only Clifford gates, then



$$\langle \psi, 0 | V^\dagger (\mathbb{I} - Z) V | \psi, 0 \rangle$$

$$= \langle \psi, 0 | g_1^\dagger \cdots g_m^\dagger (\mathbb{I} - Z) g_m \cdots g_1 | \psi, 0 \rangle$$

$$= \langle \psi, 0 | g_1^\dagger \cdots g_{m-1}^\dagger (\mathbb{I} - P_{m-1}) g_{m-1} \cdots g_1 | \psi, 0 \rangle$$

$$\vdots$$

$$= 1 - \langle \psi, 0 | P_0 | \psi, 0 \rangle$$

$$\left.\begin{aligned} P_m &= Z \\ P_i &= g_{i+1}^\dagger P_{i+1} g_{i+1} \end{aligned}\right\} \text{Paulis}$$

Pf sketch (cont.)

$$P_0 = P_0^{(1)} \otimes \dots \otimes P_0^{(n)}$$

$$\max_{|\psi\rangle} 1 - \langle \psi, 0 | P_0 | \psi, 0 \rangle = 1 - \langle 0 | P_0' | 0 \rangle$$

no dependence on $|\psi\rangle$!

$P_m = Z$ from prev. slide

$P_i = g_{i+1}^{\dagger} P_{i+1} g_{i+1}$

What happens if gate $g_i = T$? (only non-Clifford)

$T^{\dagger} Z T = T, \quad T^{\dagger} \mathbb{I} T = \mathbb{I},$

$T^{\dagger} X T = \frac{1}{\sqrt{2}} X - \frac{1}{\sqrt{2}} Y, \quad T^{\dagger} Y T = \frac{1}{\sqrt{2}} X + \frac{1}{\sqrt{2}} Y.$

Then $P_i = P_i^{(1)} + P_i^{(2)}$ ⟵ sum of 2 Pauli's.

Continue propogation till end where we reach $P_0^{(1)} + P_0^{(2)}$.

Pf sketch (cont.)

Easily extends to t T-gates to show that problem equiv. to

$$\max_{|\psi\rangle} \langle \psi, 0| V^\dagger (\mathbb{I} - Z) V |\psi, 0\rangle = \max_{|\psi\rangle} \langle \psi, 0| \sum_{i=1}^{\leq 2^t} P_0^{(i)} |\psi, 0\rangle .$$

Issue: Paulis $P_0^{(i)}$ are n qubit Paulis. Still large sum.

Ans: $\exists$ basis of $t+1$ Paulis s.t. each Pauli $P_0^{(i)}$ can be expressed as prod of basis terms.

Pf Induction with each T-gate.

$\exists$ Clifford rotation s.t. basis is mapped onto $t+1$ qubits.

Pf sketch (cont.)

Basis $B_1, \ldots, B_{t+1}$. Then construct map

$$B_1 \longrightarrow Z I I \ldots$$

$$B_2 \longrightarrow \begin{cases} I Z I I \ldots & \text{if } B_1, B_2 \text{ commute} \\ X Z I I \ldots & \text{if not} \end{cases}$$

$$B_i \longrightarrow X^{[B_1 B_i = -B_i B_1]} \cdots X^{[B_{i-1} B_i = -B_i B_{i-1}]} Z I I \ldots$$

$\Rightarrow$ Every $P_0^{(i)}$ acts on $t+1$ qubits.

Can improve to t qubits and explicit map W (see paper).

Final thoughts before I finish

1. Devote more research to understanding descriptions of q. states. Not the same as decision problems!

2. Simpler descriptions lead to decision problem speedups.

3. Big open questions are

   3a. $QCMA \stackrel{?}{=} QMA$

   3b. Is description complexity robust to small perturbations? NLTS conjecture.