Online Bidding Algorithms for Return-on-Spend Constrained Advertisers

Joint work with
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Problem Introduction

- Online advertising: a multi-billion dollar industry
- Emergence of optimization algorithms for bidding
- This talk: value maximization for the single bidder under the return-on-spend and fixed budget constraints
Problem Setup: Online Bidding under RoS Constraint

We study online bidding for a single agent. At each time step $t$, ...
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Myerson
Problem Setup: Online Bidding under RoS Constraint

We study online bidding for a single agent. At each time step $t$,

- the agent sees an ad query with its associated value $v_t$,
- submits their bid $b_t$,
- and then sees the allocation $x_t(b_t)$ and price $p_t(b_t)$.

Myerson (1981)

truthful auction

$$p_t(b) = b \cdot x_t(b) - \int_{z=0}^{b} x_t(z) \, dz$$
Problem Setup: Formalized

Our goal is to solve, with minimum regret and minimum constraint violation, the online bidding problem:

\[
\begin{align*}
\text{maximize} & \quad \sum_{t=1}^{T} v_t \cdot x_t(b_t) \\
\text{subject to} & \quad \sum_{t=1}^{T} p_t(b_t) \leq \sum_{t=1}^{T} v_t \cdot x_t(b_t), \\
& \quad \sum_{t=1}^{T} p_t(b_t) \leq \rho T.
\end{align*}
\]

RoS constraint

Model first proposed by Mannor, Tsitsiklis, Yu. The RoS constraint is non-packing; hence, inapplicability of:

- bandits-with-knapsacks (e.g., Immorlica, Sankararaman, Schapire, Slivkins)
- allocation-under-resource-consumption (e.g., Balseiro, Lu, Mirrokni).
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Problem Setup: Formalized

Our goal is to solve, with minimum regret and minimum constraint violation, the online bidding problem:

maximize \( \sum_{t=1}^{T} v_t \cdot x_t(b_t) \)
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\( \sum_{t=1}^{T} p_t(b_t) \leq \rho T \).

We define regret relative to the best adaptive strategy in hindsight:

\[
\text{Regret}(\text{Alg}, \overrightarrow{\gamma}) := \mathbb{E}_{\gamma \sim \mathcal{P}_T} \left[ \text{Reward}(\text{Opt}, \overrightarrow{\gamma}) - \text{Reward}(\text{Alg}, \overrightarrow{\gamma}) \right].
\]
Our Main Result

Theorem 1: Our Main Result (Informal)

For a $T$-length input i.i.d. sequence of ad queries, our algorithm provably attains, under a mild technical assumption, $O(\sqrt{T \log T})$ regret while respecting both the budget and RoS constraints.
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- First algorithm to attain near-optimal regret while satisfying both budget and RoS constraints in any outcome.

- In doing so, we improve upon the prior work of Balseiro, Lu, and Mirrokni (2020), which obtains similar guarantees under only budget constraints.
Related Work

- Balseiro, Lu, Mirrokni (2020): fixed-budget constraints
- Castiglioni, Celli, Marchesi, Romano, Gatti (2022): more general, with weaker guarantees for RoS
- Agrawal and Devanur (2014): requires bounded dual space
- Golrezaei, Jaillet, Liang, Mirrokni (2021): constraints hold in expectation

Other related work:
- AdWords: Mehta, Saberi, Vazirani, Vazirani (2007)
Our Techniques
**Our Algorithmic Outline: RoS Constraints**

We first solve the problem with only the RoS constraint:

\[
\begin{align*}
\text{maximize} & \quad \sum_{t=1}^{T} v_t \cdot x_t(b_t) \\
\text{subject to} & \quad \sum_{t=1}^{T} p_t(b_t) \leq \sum_{t=1}^{T} v_t \cdot x_t(b_t).
\end{align*}
\]

We use the **primal-dual framework** similar to that in Balseiro, Lu, and Mirrokni (2020), which lends our algorithm adaptivity to changing input values, with **Online Mirror Descent (OMD)** to update the dual variable (which tracks the constraint violation).
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\end{align*}
\]

Our update rule solves

\[
\begin{align*}
\text{maximize}_{\{b_i\}} & \quad \left\{ \sum_{i=1}^{T} v_i \cdot x_i(b_i) + \min_{\lambda \geq 0} \left[ \lambda \cdot \sum_{i=1}^{T} g_i(b_i) + \frac{1}{\alpha} h(\lambda) \right] \right\}
\end{align*}
\]

where

- \( g_i(b) := v_i \cdot x_i(b) - p_i(b) \) measures the constraint satisfaction, and
- \( h \) is generalized negative entropy, which imposes a large (exponential) penalty on constraint violation.
Our Algorithm’s Updates: Approximate RoS Constraints

\[
\text{maximize}_{\{b_i\}} \left\{ \sum_{i=1}^{T} v_i \cdot x_i(b_i) + \min_{\lambda \geq 0} \left[ \lambda \cdot \sum_{i=1}^{T} g_i(b_i) + \frac{1}{\alpha} h(\lambda) \right] \right\}
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- Bid \( b_t \) maximizes the current penalty-adjusted reward:

\[
b_t = \arg \max_{b \geq 0} \left[ \frac{1 + \lambda_t}{\lambda_t} \cdot v_t \cdot x_t(b) - p_t(b) \right] = v_t + \frac{v_t}{\lambda_t}.
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\]

- Dual variable \( \lambda_t \) penalizes (rewards) constraint violation (satisfaction):

\[
\lambda_{t+1} = \arg \min_{\lambda \geq 0} \left[ g_t(b_t) \cdot \lambda + \frac{1}{\alpha} V_h(\lambda, \lambda_t) \right] = \lambda_1 \cdot \exp \left[ - \sum_{i \leq t} \alpha \cdot g_i(b_i) \right]
\]
Our Algorithm: Approximate RoS Constraints

**Input.** Time horizon $T$ and ad requests $\gamma \sim \mathcal{P}^T$.

**Initialize** step size $T^{-1/2}$ and dual variable $\lambda_1 = 1$. 
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- Observe the value $v_t$, and set the bid $b_t = v_t + \frac{v_t}{\lambda_t}$.
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- Observe the value $v_t$, and set the bid $b_t = v_t + \frac{v_t}{\lambda_t}$.
- Observe the price $p_t(b_t)$ and allocation $x_t(b_t)$

Return the sequence $\{b_t\}_{t=1}^T$ of generated bids.
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**Input.** Time horizon $T$ and ad requests $\overrightarrow{\gamma} \sim \mathcal{P}^T$.

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- Compute the gradient $g_t(b_t) = v_t \cdot x_t(b_t) - p_t(b_t)$.

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- Update the dual variable as $\lambda_{t+1} = \lambda_t \cdot \exp \left[ -\alpha \cdot g_t(b_t) \right]$.

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**Return** the sequence $\{b_t\}_{t=1}^T$ of generated bids.
Theorem 2: RoS Constraint Violation Bound (Informal)

Our algorithm's RoS constraint violation at most $O(\sqrt{T} \log T)$. 

- The proof idea is as follows.
- When the cumulative violation $\sum_{i \leq t} g_i(b_i) \gg \alpha - 1$, the update rule $\lambda_{t+1} = \lambda_1 \cdot \exp(-\alpha \sum_{i \leq t} g_i(b_i))$ makes $\lambda_{t+1}$ huge,
- and as a result, the update $b_{t+1} = v_{t+1} + v_{t+1} \lambda_{t+1}$ prevents us from overbidding.
Bound on Approximate RoS Violation

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- ... and as a result, the update \( b_{t+1} = v_{t+1} + \frac{v_{t+1}}{\lambda_{t+1}} \) prevents us from overbidding.
Bound on Regret

The primal-dual framework implies

$$\text{Regret}(\text{Alg}, \vec{\gamma}) \leq \mathbb{E}_{\vec{\gamma} \sim \mathcal{P}_T} \left[ \sum_{t \in [T]} \lambda_t \cdot g_t(b_t) \right].$$
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**Bounding the R.H.S.** requires novel analysis due to the existence of input instances for which $\lambda_t$ can be huge.

different from fixed budget setting
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Bounding the R.H.S. requires novel analysis due to the existence of input instances for which $\lambda_t$ can be huge.

Our novel insight to bound $\sum_{t \leq T} \lambda_t \cdot g_t(b_t)$ is to combine:

- structural properties of the gradient
- with a white-box OMD analysis
- and tools developed by Allen-Zhu and Orecchia (2015) for solving positive linear programs.
Towards a Regret Bound: Bounding $\sum_{t \leq T} \lambda_t \cdot g_t$

Theorem 3: Key Regret Bound Lemma (Informal)

Our algorithm's iterates satisfy $\sum_{t \leq T} g_t \cdot \lambda_t \leq O(\sqrt{T})$. 

Proof. We have, adding/subtracting $\alpha g_t \lambda_t$ dual update step introducing Bregman divergence local strong convexity Allen-Zhu & Orecchia completing the square
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$$\alpha g_t \lambda_t = \alpha g_t (\lambda_t - \lambda_{t+1}) + \alpha g_t \lambda_{t+1}$$

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$$= (1 + \alpha)(\lambda_t - \lambda_{t+1}) - \mathcal{V}^h_{\lambda_t}(\lambda_{t+1})$$

introducing Bregman divergence
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= (1 + \alpha)(\lambda_t - \lambda_{t+1}) - \mathcal{V}_{\lambda_t}^h (\lambda_{t+1}) \\
\leq (1 + \alpha)(\lambda_t - \lambda_{t+1}) - \frac{(\lambda_t - \lambda_{t+1})^2}{2 \max(\lambda_t, \lambda_{t+1})}
$$

local strong convexity

Allen-Zhu & Orecchia
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$$= (1 + \alpha)(\lambda_t - \lambda_{t+1}) - \mathcal{V}_{\lambda_t}^h(\lambda_{t+1})$$

$$\leq (1 + \alpha)(\lambda_t - \lambda_{t+1}) - \frac{(\lambda_t - \lambda_{t+1})^2}{2 \max(\lambda_t, \lambda_{t+1})}$$

$$\leq \lambda_t - \lambda_{t+1} + \frac{1}{2} \alpha^2 g_t^2 \max(\lambda_t, \lambda_{t+1})$$

We now provide an upper bound on $\frac{1}{2} \alpha^2 g_t^2 \max(\lambda_t, \lambda_{t+1})$. 
Regret Bound Continued: Bounding $\frac{1}{2} \alpha^2 g_t^2 \max(\lambda_t, \lambda_{t+1})$
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**Lemma 1**

Our gradients $g_t$ satisfy the bound $\max(-1, -1/\lambda_t) \leq g_t \leq v_t x_t$
Regret Bound Continued: Bounding $\frac{1}{2} \alpha^2 g_t^2 \max(\lambda_t, \lambda_{t+1})$

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Case 1. Assume $g_t \geq 0$. Then,
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$g_t \leq 1$, $\alpha = T^{-1/2}$

$\exp(-x) \leq 1 - x/2$ for $x \in [0, 1.5]$
Regret Bound Continued: Bounding $\frac{1}{2} \alpha^2 g_t^2 \max(\lambda_t, \lambda_{t+1})$

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dual update rule
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$$\frac{1}{2} \alpha^2 g_t^2 \max(\lambda_t, \lambda_{t+1}) = \frac{1}{2} \alpha^2 g_t^2 \lambda_t \leq \alpha (\lambda_t - \lambda_t \exp(-\alpha g_t)) \leq \alpha (\lambda_t - \lambda_{t+1})$$

**Case 2**. Assume $g_t \leq 0$. Then,

$$\frac{1}{2} \alpha^2 g_t^2 \max(\lambda_t, \lambda_{t+1}) = \frac{1}{2} \alpha^2 g_t^2 \lambda_{t+1}$$
Regret Bound Continued: Bounding $\frac{1}{2}\alpha^2 g_t^2 \max(\lambda_t, \lambda_{t+1})$

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$0 \geq g_t \geq \max(-1, -1/\lambda_t)$
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*Note: The dual update rule is applied in the final step.*
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$g_t \geq -1, \alpha = T^{-1/2}$
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Therefore, we have $\frac{1}{2} \alpha^2 g_t^2 \max(\lambda_t, \lambda_{t+1}) \leq \alpha(\lambda_t - \lambda_{t+1}) + 2\alpha^2$. 
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Summing over $t \in \{T\}$ and dividing by $\alpha$ finishes the proof.
Putting It All Together

To get strict RoS satisfaction, we propose a simple idea:

- First, submit a sequence of bids so as to accumulate a slack on the RoS constraint.
- Next, run the existing algorithm, which suffers some bounded constraint violation, which is compensated by the slack from the first phase.
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To get strict RoS satisfaction, we propose a simple idea:

► First, submit a sequence of bids so as to accumulate a slack on the RoS constraint.

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To satisfy both RoS and budget constraints, we combine our ideas with those of Balseiro, Lu, and Mirrokni (2020).
Thank You!