Sparse plus low-rank graphical models of time series

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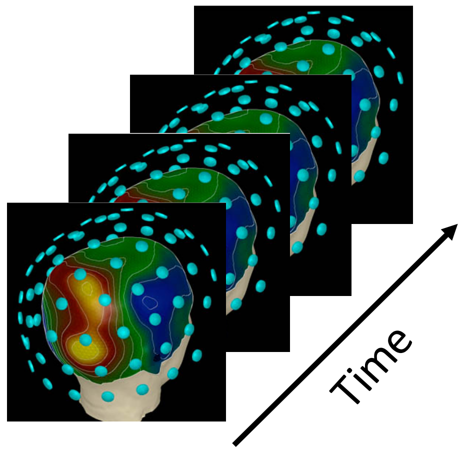
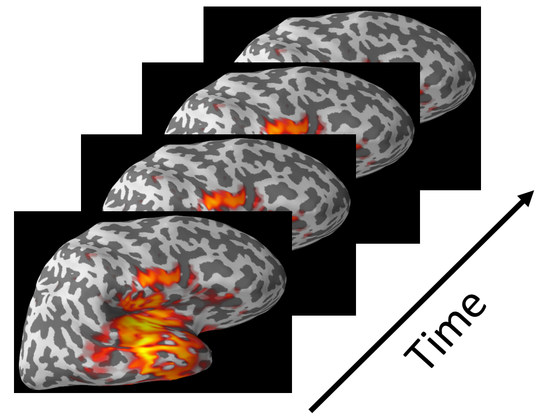
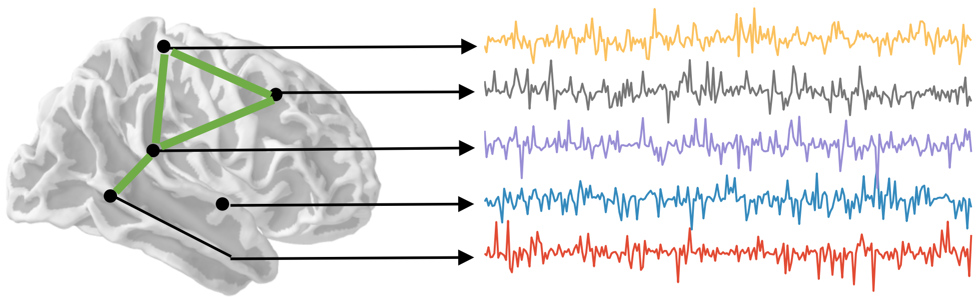
Brain Interactions from MEG

Magnetoencephalography (MEG) captures weak magnetic field.







Goal: Infer functional connectivity

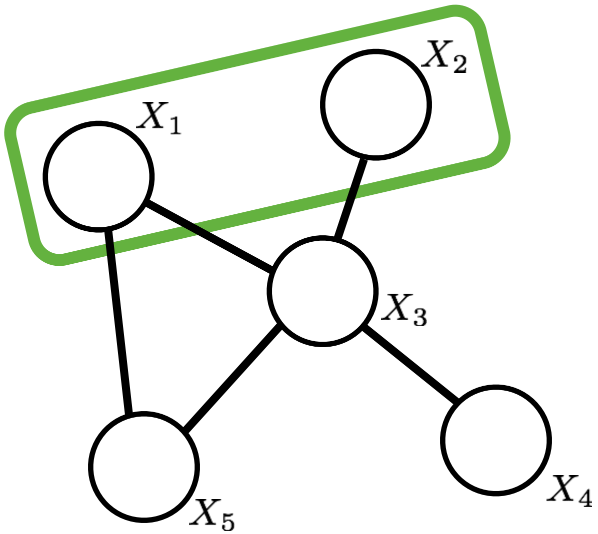
Graphical Models

- Graph G=(V, E) encodes conditional independence statements.

nodes edges

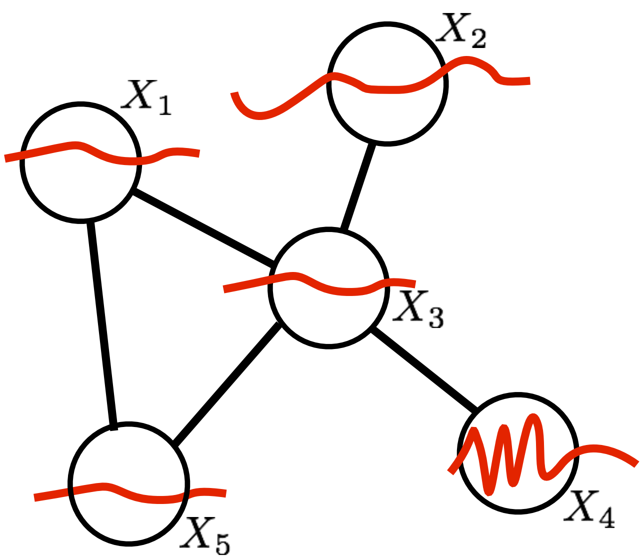
No edge (i, j) $\longleftrightarrow$ X_i , X_j conditionally independent given rest of variables.

$$X_1 \perp\!\!\!\perp X_2 | X_3, X_4, X_5$$



Graphical Models of Time Series

No edge (i,j) $\Rightarrow$ **time series** X_i , X_j conditionally independent given *entire trajectories* of other series.

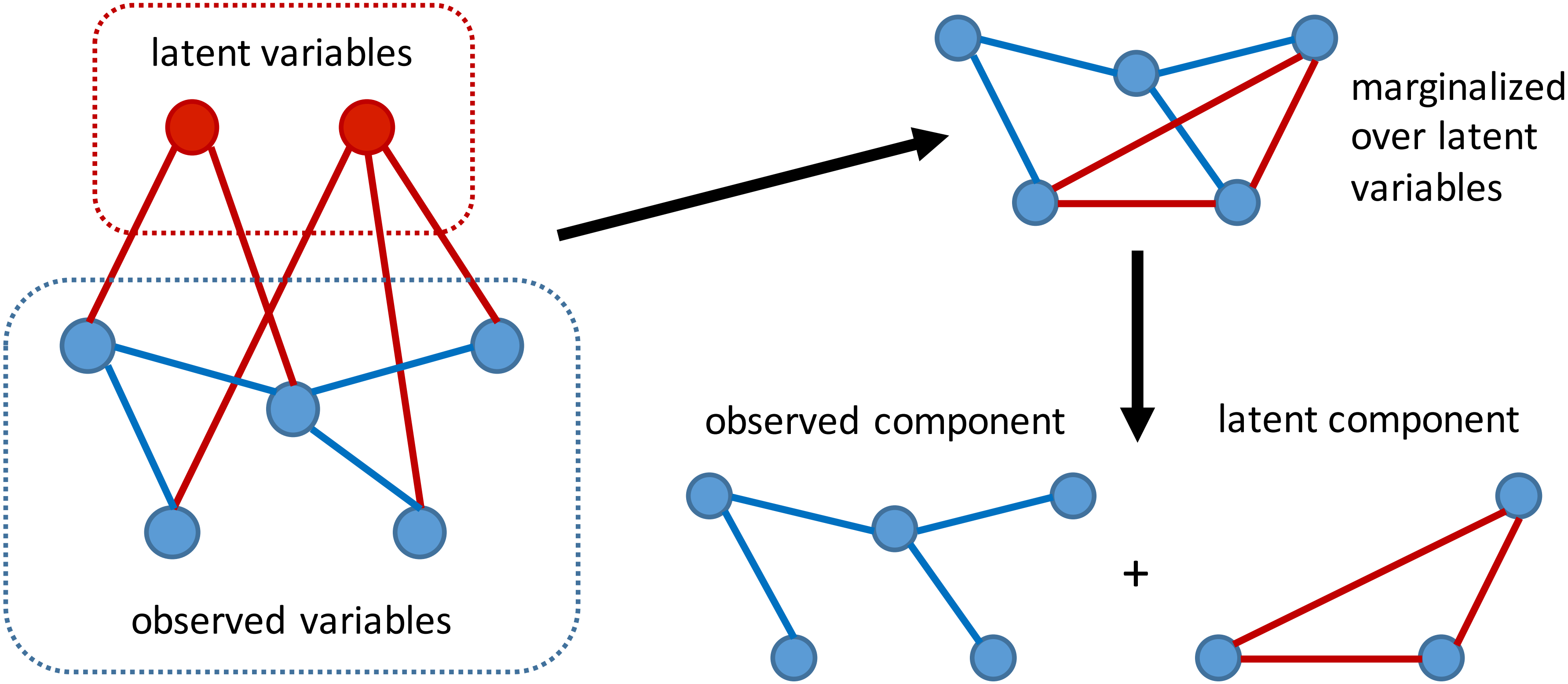


- Accounts for interactions at *all lags*.
- Removes linear effects of other series.

Natural property for **functional connectivity**

Examples of existing work:
Bach et al. 2004, Songsiri & Vandenberghe 2010, Jung et al. 2015, Tank et al. 2015

Latent structure



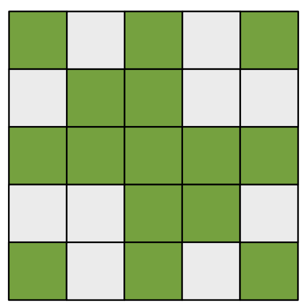
Examples of existing work:
Chandrasekaran et al. 2012, Jalali & Sanghavi 2012, Liégois et al. 2015

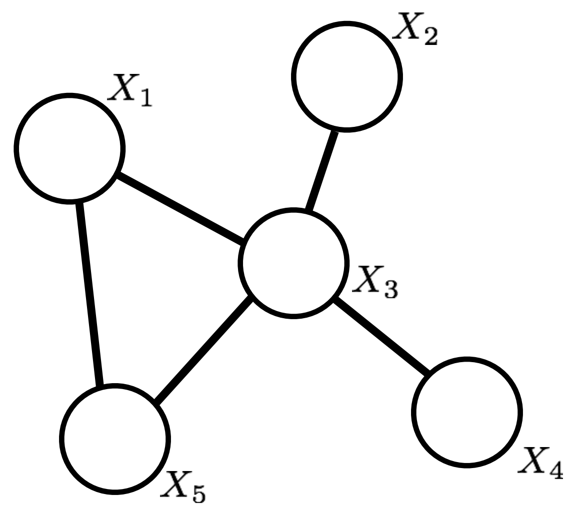
Encoding graph structure

Gaussian random vectors

- For Gaussian random vector $X \sim \mathcal{N}(0, \Sigma)$

Conditional independence encoded in the precision matrix.

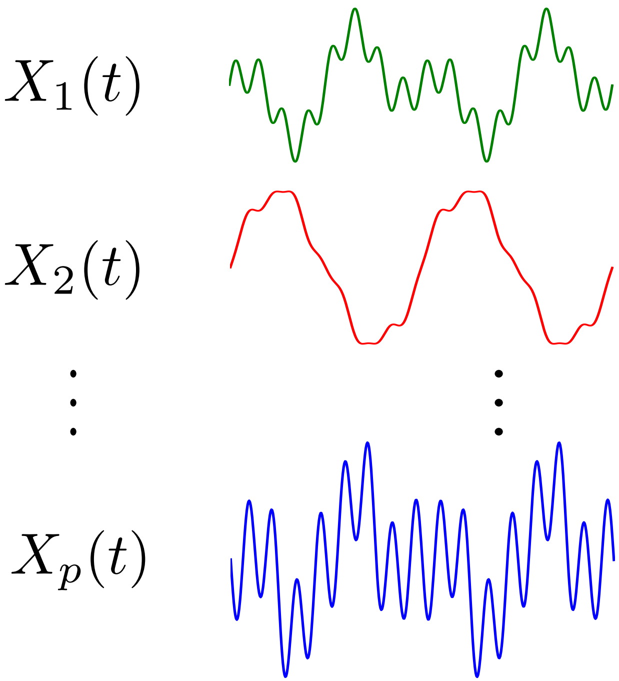
$\Sigma^{-1} =$ 



$$(\Sigma^{-1})_{ij} = 0$$
$$\Updownarrow$$

X_i , X_j conditionally independent given rest of variables.

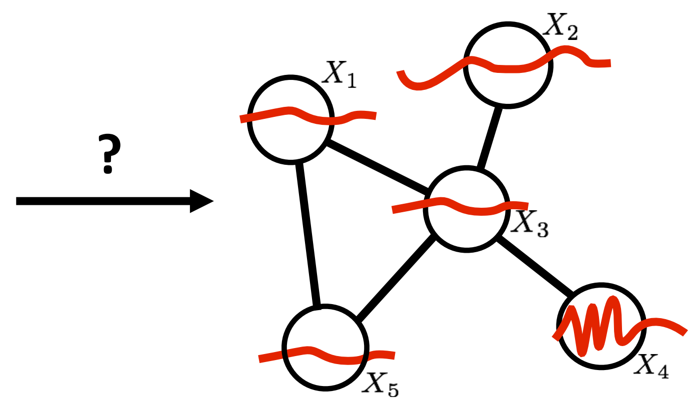
Gaussian stationary processes



lagged covariance:

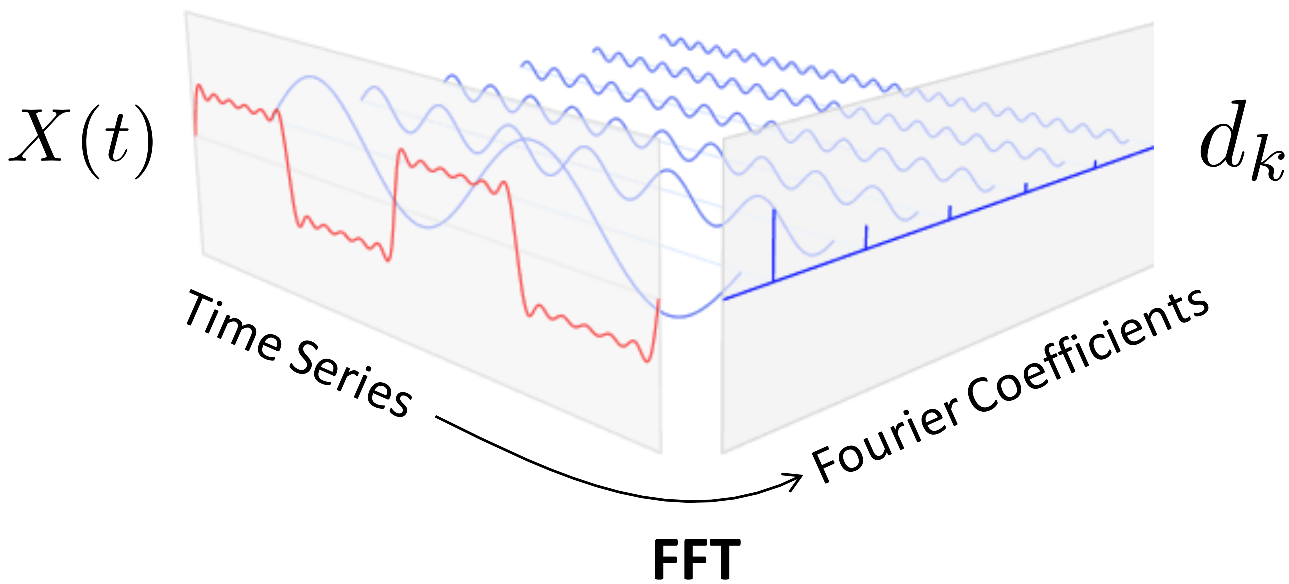
$$\Gamma(h) = \mathrm{Cov}(X(t), X(t+h))$$

$$\Gamma(h)$$



How is conditional independence encoded?

Model in the Frequency Domain



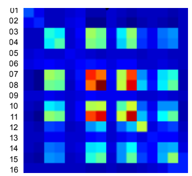
Spectral density matrix

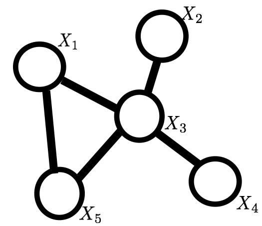
Lagged covariance matrix

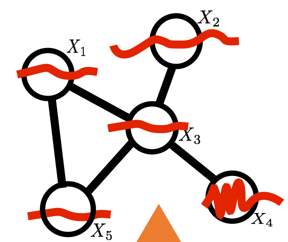
$$\Gamma(h) = \mathrm{Cov}(X(t), X(t+h)) \longrightarrow S(\lambda) = \sum_{h=-\infty}^{\infty} \Gamma(h) e^{-i\lambda h}$$

Encoding structure in frequency domain

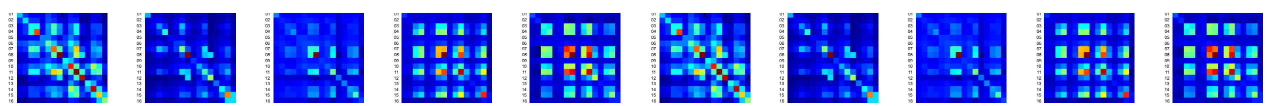
For **Gaussian** i.i.d. random variables,

$\Sigma^{-1} =$ 





(*Dahlhaus, 2000*) For **Gaussian** stationary time series,
$S(\lambda)^{-1}$:



$\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_k \quad \cdots \quad \lambda_T$

complex inverse **spectral density** matrices

Learning structure from data

Penalized likelihood expression

inverse covariance matrix

$$-\log\left(\text{Likelihood}(\Psi)\right) \quad + \quad \lambda \cdot \text{Penalty}(\Psi)$$

sample covariance matrix

$$-\log\det\Psi + \text{tr}\left\{\hat{S}\Psi\right\} \qquad \sum_{i<j}|\Psi_{ij}|$$

negative log-likelihood of Gaussian

sparsity-inducing penalty

Graphical LASSO (Friedman et al. 2007)

solved with: many existing algorithms

What's our likelihood in the frequency-domain case?

Likelihood in the Frequency Domain

Time Domain Likelihood

$$p(X(1),\ldots,X(T)|[\Gamma(h)]_{h=0}^{T-1})$$

$\longrightarrow$

Frequency Domain Likelihood

$$p(\underbrace{d_0,\ldots,d_{T-1}}_{\text{Fourier coefficients}}|\{S(\lambda_k)\}_{k=0}^{T-1})$$

Fourier coefficients are asymptotically independent, complex Normal random vectors (Brillinger, 1981)

Whittle Approximation

$$S_k \equiv S(\lambda_k)$$

$$p(d_1,\ldots,d_T|\{S(\lambda_k)\}_{k=0}^{T-1}) \approx \prod_{k=0}^{T-1} \frac{1}{\pi^p|S_k|} e^{-d_k^* S_k^{-1} d_k}$$

Penalized likelihood expression in frequency domain

inverse spectral density matrix

$$-\log\left(\prod_{k=0}^{T-1}\frac{1}{\pi^p|S_k|}e^{-d_k^* S_k^{-1} d_k}\right) \quad + \quad \lambda\cdot\mathrm{Penalty}(\Psi)$$

sample spectral density matrix

$$\sum_{k=0}^{T-1}\Big(-\log\det\Psi[k]+\mathrm{tr}\left\{\hat{S}[k]\Psi[k]\right\}\Big) \qquad \sum_{i<j}\sqrt{\sum_{k=0}^{T-1}|\Psi[k]_{ij}|^2}$$

Group LASSO penalty

Spectral graphical LASSO (Jung et al. 2015)

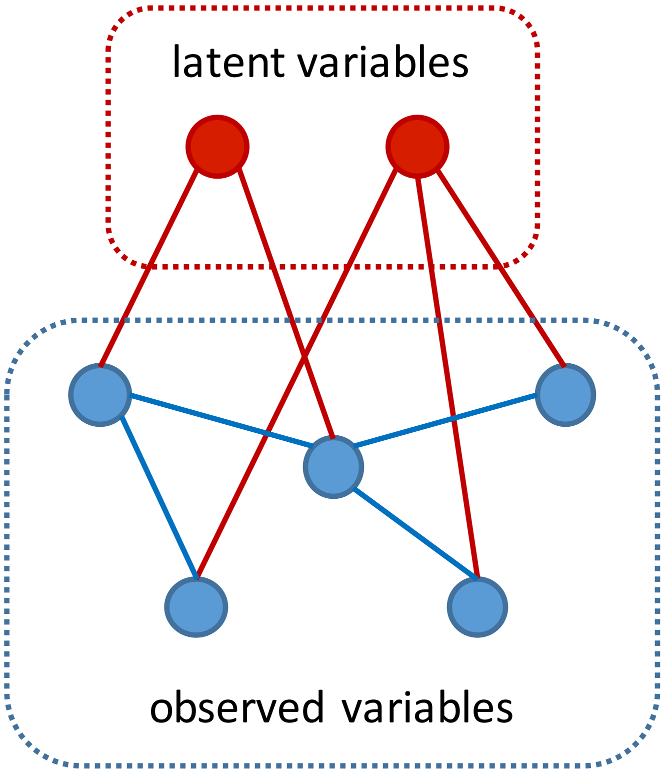
solved with: ADMM (Jung et al. 2015)

Incorporating latent processes

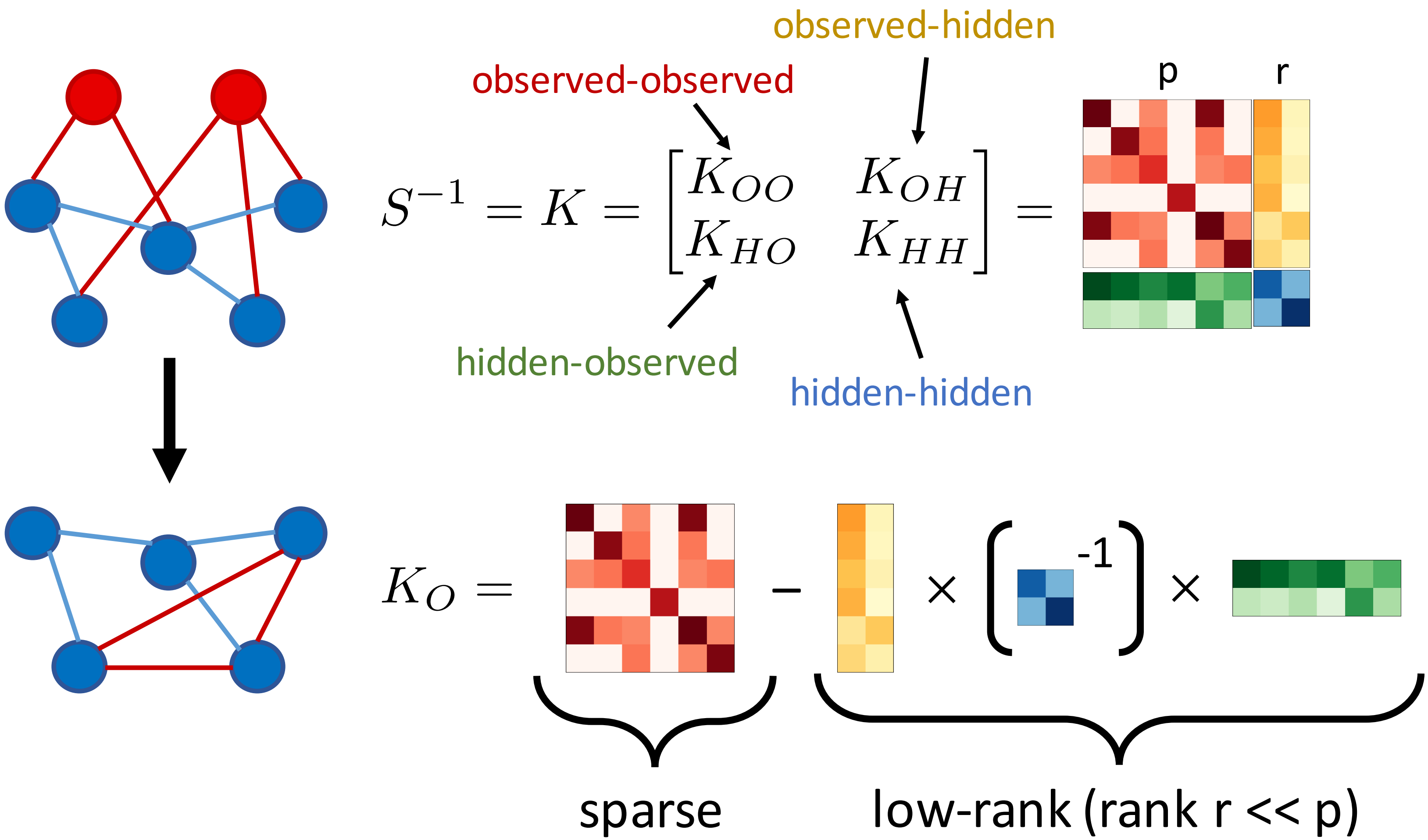
Latent structure in MEG

- MEG recordings affected by neural activity unrelated to task
- Mapping from recordings to brain activity introduces "point spread"

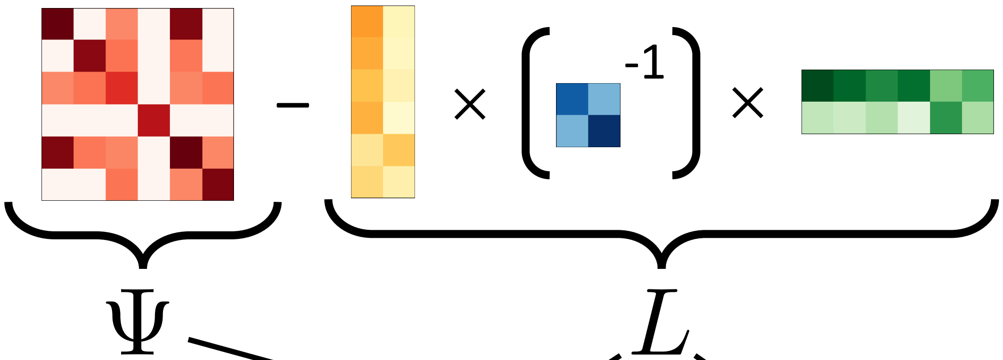
These issues can be addressed by adding a latent component to the model



Sparse plus low-rank decomposition



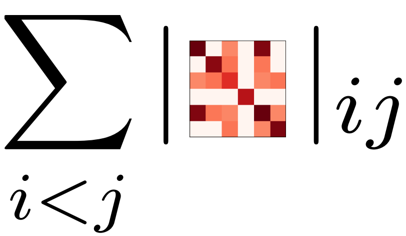
Sparse plus low-rank penalized likelihood



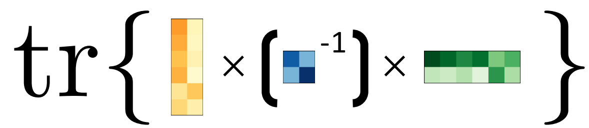
negative log-likelihood:

$$-\log\det(\Psi - L) + \mathrm{tr}\left\{\hat{S}(\Psi - L)\right\}$$

sparse penalty:



low-rank penalty:



Latent variable GLASSO (Chandrasekaran et al. 2012)

solved with ADMM (Ma et al. 2013)

Latent variable spectral GLASSO

negative log-likelihood:

$$\sum_{k=0}^{T-1} \left(-\log\det(\Psi[k] - L[k]) + \mathrm{tr}\left\{ \hat{S}[k](\Psi[k] - L[k]) \right\} \right)$$

Whittle approximation

sparse penalty:

$$\sum_{i<j} \sqrt{\sum_{k=0}^{T-1} |\Psi[k]_{ij}|^2}$$

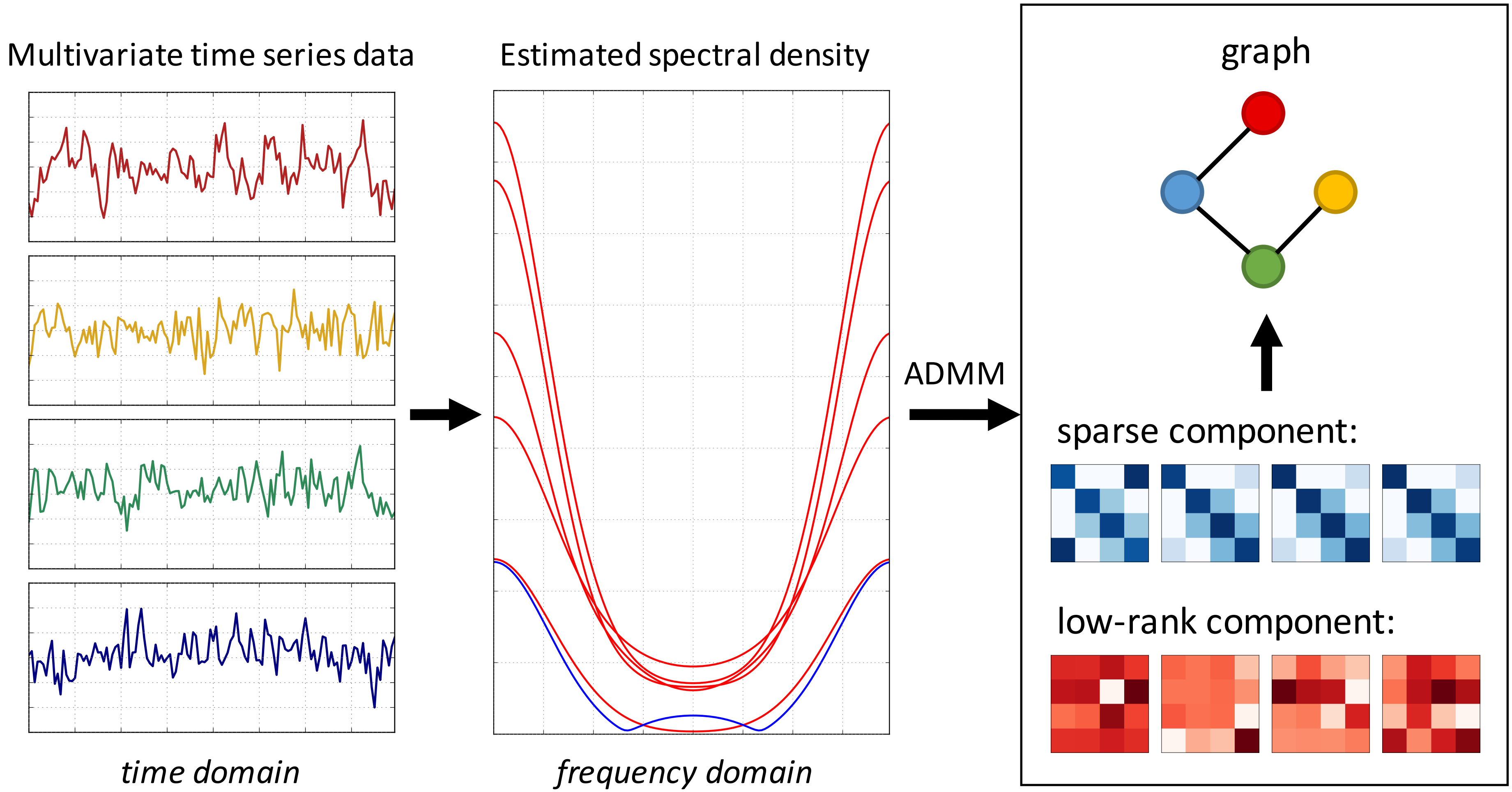
Group LASSO penalty

low-rank penalty:

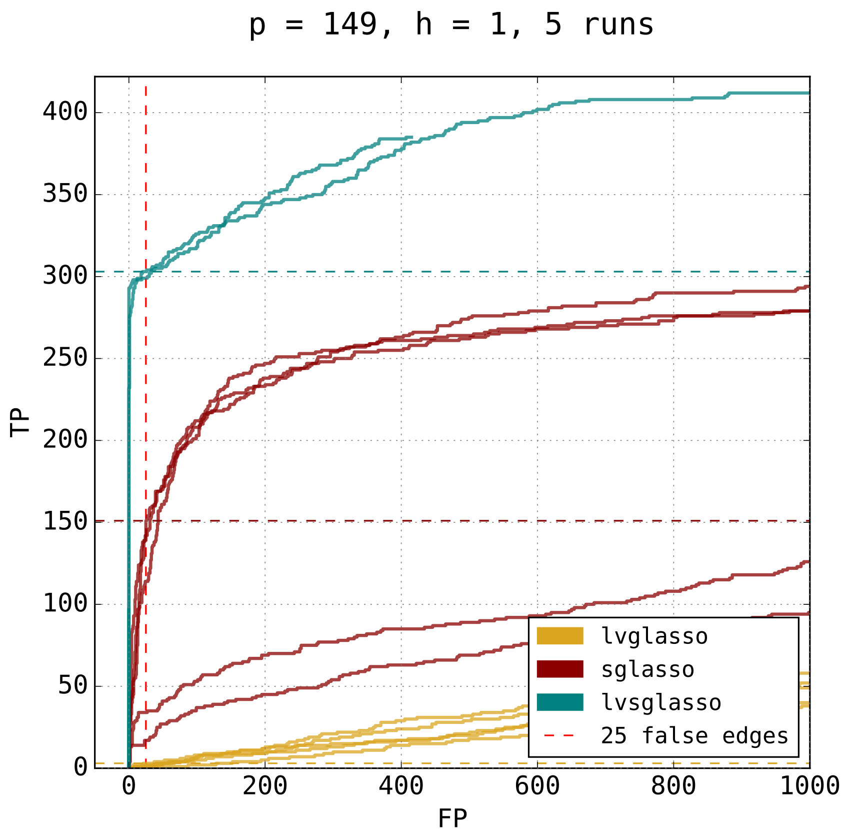
$$\sum_{k=0}^{T-1} \mathrm{tr}\left\{ L[k] \right\}$$

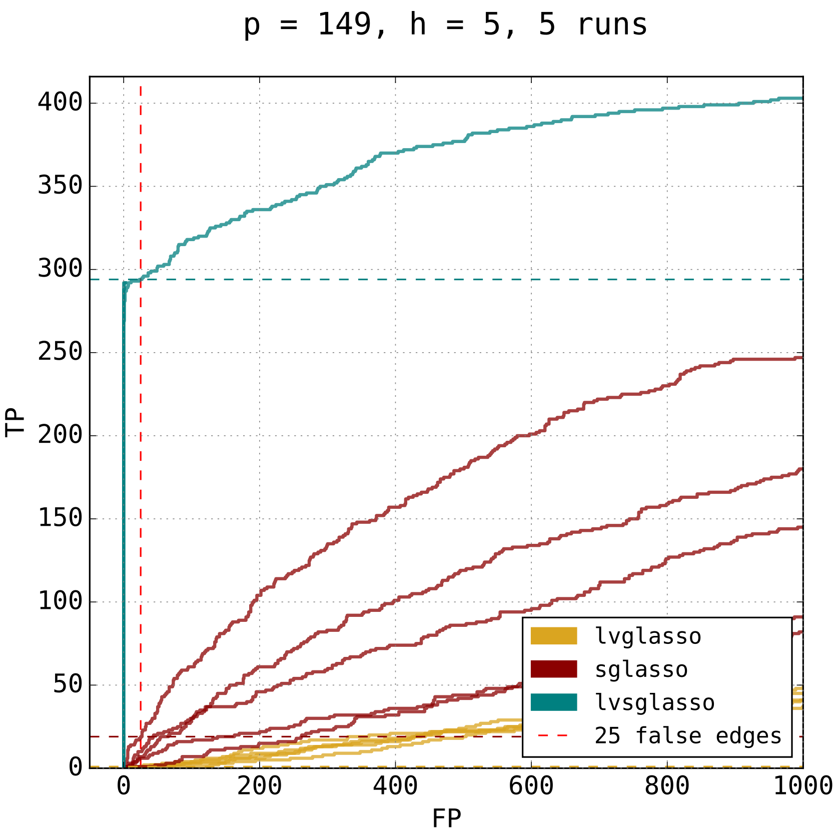
Used ADMM to solve this convex formulation

Analysis pipeline



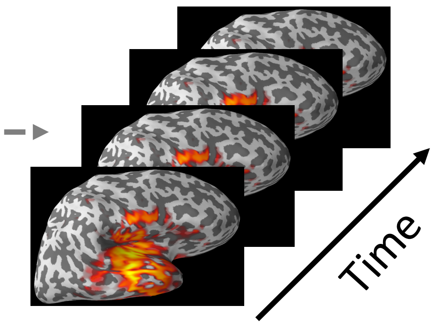
Synthetic data results



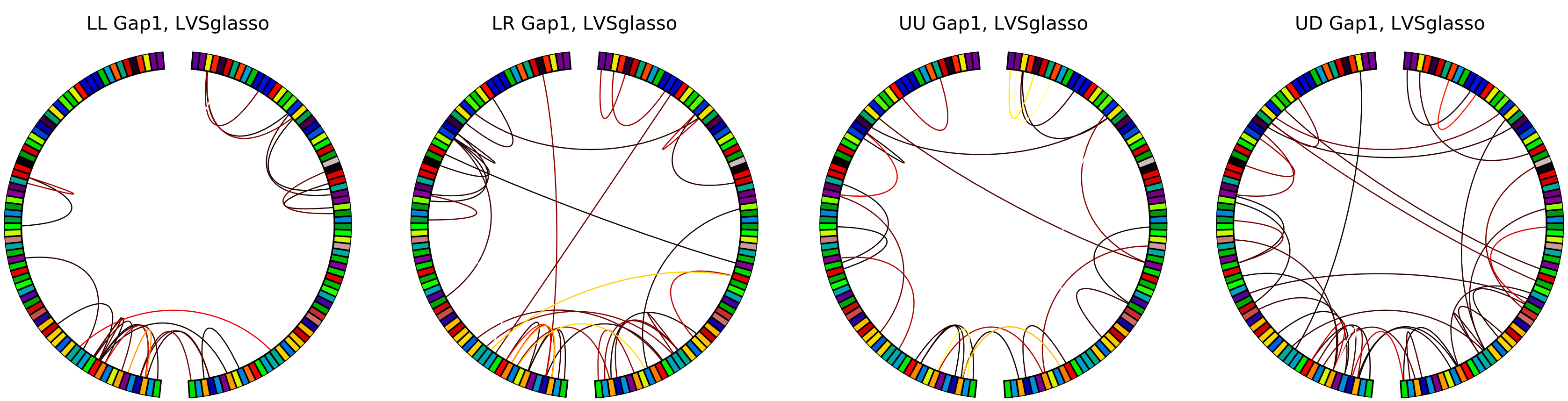


MEG Auditory Attention Analysis

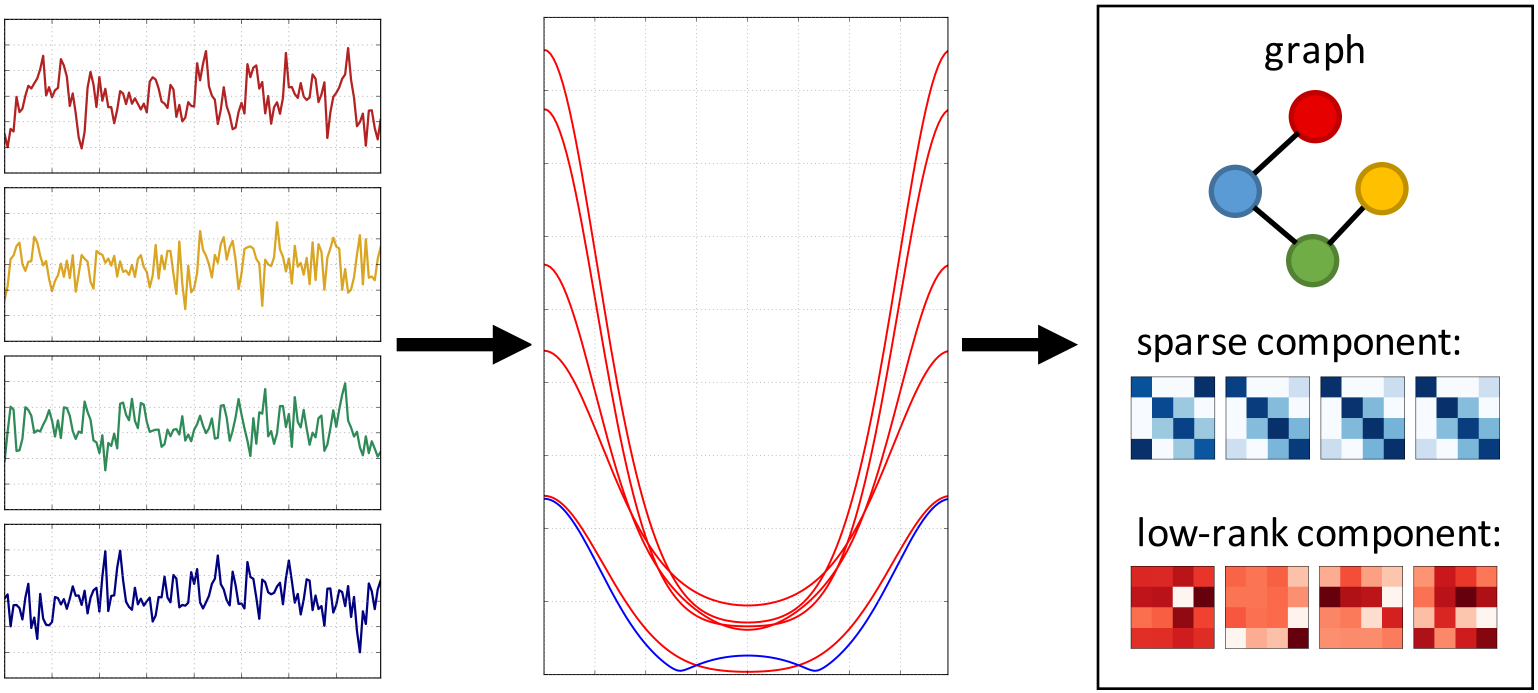
Maintain or Switch attention
(Left/Right, High/Low pitch)



- 16 subjects, 10-50 trials each.
- Each trial results in a 149-dimensional time series.



Summary



- Frequency domain for conditional independence structure and likelihood
- Modeling latent component gives sparser, more interpretable graphs
- Latent variable, spectral models are important in neuroscience