

Lecture 10 & 11: Real Stable and Hyperbolic Polynomials

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In this lecture we introduce hyperbolic and real stable polynomials as natural generalization of real-rootedness to multivariate polynomials. Then, we define strongly Rayleigh distributions as probability distribution whose *generating function* is a real stable polynomial. We will see that this is a generalization of random spanning tree distributions and determinantal distributions. We will finish this lecture by showing that strongly Rayleigh distributions satisfy strongest form negative dependence properties including negative association, stochastic dominance and log concavity.

In the next lecture we will use the properties of strongly Rayleigh measures (extending the properties of random spanning tree distributions) to design a randomized rounding approximating algorithm for the symmetric traveling salesman problem. The material of this lecture are mostly based on [BB10; BBL09; Pem13; Vis13]. We use the notation $\mathbb{R}[z_1, \dots, z_d]$ to denote a degree d polynomial in z_1, \dots, z_d with real coefficients.

10.1 Real-rooted Polynomials

We start by recalling some properties of real-rooted polynomials.

In the following simple lemma we show that imaginary roots of univariate polynomials come in conjugate pairs.

Lemma 10.1. *For any $p \in \mathbb{R}[t]$, if $p(a + \imath b) = 0$, then $p(a - \imath b) = 0$.*

In this note we typically use letter t for the univariate polynomials and z for multivariate polynomials.

Proof. Say $p(t) = \sum_{i=0}^d a_i t^{d-i}$. Then,

$$\begin{aligned}
 p(a + \imath b) = 0 &= \overline{p(a + \imath b)} \\
 &= \sum_{i=0}^d a_i \overline{(a + \imath b)^i} \\
 &= \sum_{i=0}^d a_i \overline{(a + \imath b)}^i \\
 &= \sum_{i=0}^d a_i (a - \imath b)^i = p(a - \imath b).
 \end{aligned}$$

□

There is a close connection between the real-rooted polynomials and the sum of Bernoulli random variables.

Given a probability distribution μ , over $[d] = \{0, 1, \dots, d\}$ where $\mathbb{P}_\mu[i] = a_i$. Let

$$p(t) = \sum_{i=0}^d a_i t^i.$$

We show that $p(t)$ is real-rooted if and only if μ can be written as the sum of independent Bernoulli random variables.

Suppose $p(t)$ is real-rooted. Since the coefficients of $p(t)$ are non-negative, the roots of $p(z)$ are non-positive. Therefore, we can write

$$p(t) = a_d \prod_{i=1}^d (t + \alpha_i),$$

where $\alpha_1, \dots, \alpha_d \geq 0$. Since $p(1) = 1$,

$$\frac{1}{a_d} = \prod_{i=1}^d (1 + \alpha_i),$$

Now, let $q_i = \frac{1}{1+\alpha_i}$, or equivalently, $\alpha_i = \frac{1-q_i}{q_i}$. Then,

$$\begin{aligned} p(t) &= \frac{1}{\prod_{i=1}^d (1 + \alpha_i)} \prod_{i=1}^d \left(t + \frac{1 - q_i}{q_i} \right) \\ &= \frac{1}{\prod_{i=1}^d \frac{1}{q_i}} \sum_{S \subseteq [d]} t^{|S|} \prod_{i \notin S} \frac{1 - q_i}{q_i} \\ &= \sum_{S \subseteq [d]} t^{|S|} \prod_{i \in S} q_i \prod_{i \notin S} (1 - q_i) \\ &= \sum_{k=0}^d t^k \sum_{S: |S|=k} \prod_{i \in S} q_i \prod_{i \notin S} (1 - q_i). \end{aligned}$$

Say μ is the distribution of d Bernoulli random variables where the i -th one has success probability q_i . Then the probability that exactly k of them occur is exactly equal to a_k . In other words, the real-rooted polynomials with non-negative coefficients are the same as distribution of independent Bernoulli random variables. Consequently, we can use Chernoff types of bound to show that these a_i 's which are far from the expectation $\sum_{k=0}^d k \cdot a_k$ are very small. Next we show that these sequence are highly concentrated by means of ultra log concavity.

Definition 10.2 (Log Concavity). A sequence $\{a_0, a_1, \dots, a_d\}$ of nonnegative numbers is said to be log-concave if for all $0 < k < d$,

$$a_{k-1} \cdot a_{k+1} \leq a_k^2.$$

We say the sequence is ultra log concave if for all $0 < k < d$,

$$\frac{a_{k-1}}{\binom{d}{k-1}} \cdot \frac{a_{k+1}}{\binom{d}{k+1}} \leq \left(\frac{a_k}{\binom{d}{k}} \right)^2.$$

Note that any log concave sequence of nonnegative numbers is unimodal, i.e., there is a number k such that

$$\dots \leq a_{k-2} \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \dots$$

Next, we want to show that the coefficients of a real-rooted degree d polynomial with nonnegative coefficients are ultra log concave. This is also known as the Newton inequalities.

Using the above argument this implies that a sum of independent Bernoulli random variables is a log-concave probability distribution, that is if a_i is the probability that exactly i of them occur, then $a_{i-1} \cdot a_{i+1} \leq a_i^2$ for all i . Consequently, any such distribution is unimodal.

10.1.1 Closure Properties of Real-rooted Polynomials

In this part we show that the coefficients of a real rooted polynomial with nonnegative coefficients are (ultra) log concave. The proof uses closure properties of real-rootedness. We start by describing these properties and then we prove the claim.

Given a real-rooted polynomial, usually it is a non-trivial task to verify real-rootedness without actually computing the roots. In the next section we will see how the extension of real-rootedness to multivariate polynomials can help us with this task.

One way to verify real-rootedness of a polynomial $p(t)$ is to start from a polynomial $q(t)$ that is real-rooted and then use a real-rooted preserving operator to derive $p(t)$ from $q(t)$. Now, let us study some basic operations that preserve real-rootedness.

- i) If $p(t)$ is real-rooted then so is $p(c \cdot t)$ for any $c \in \mathbb{R}$.
- ii) If $p(t)$ is a degree d real-rooted polynomial then so is $t^d p(1/t)$.
- iii) If $p(t)$ is real-rooted then so is $p'(t)$. This basically follows from the fact that there is a root of p' between any consecutive roots of p there is exactly one root of p' . In other words, the roots of p' *interlace* the roots of p . See the next subsection for more details.

The last property is perhaps the most non-trivial one; it is a special case of the Gauss-Lucas theorem.

Theorem 10.3 (Gauss-Lucas). *For any polynomial $p \in \mathbb{C}[t]$, the roots of p' can be written as a convex combination of the roots of p .*

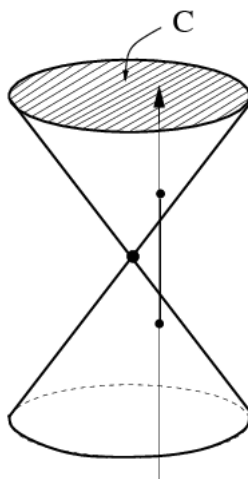
Lemma 10.4 (Newton Inequalities). *For any real-rooted polynomial $p(t) = \sum_{i=0}^d a_i t^i$, if $a_0, \dots, a_k \geq 0$, then it is ultra log concave.*

Proof. We apply the close properties a number of times. First, by the closure of the derivative $p_1(t) = \frac{d^{i-1}}{dt^{i-1}} p(t)$ is real-rooted. This shaves off all of the coefficients a_0, \dots, a_{i-2} . By (ii), $p_2(t) = t^{d-i+1} p_1(1/t)$ is real rooted. This reverse the coefficients. By (iii) $p_3(t) = \frac{d^{d-i-1}}{dt^{d-i-1}} p_2(t)$ is real-rooted. This shaves off all of the coefficients a_{i+2}, \dots, a_d . So, $p_3(t)$ is a degree 2 real-rooted polynomial,

$$p_3(t) = \frac{d!}{2} \left(\frac{a_{i-1}}{\binom{d}{i-1}} t^2 + \frac{2a_i}{\binom{d}{i}} t + \frac{a_{i+1}}{\binom{d}{i+1}} \right).$$

The above polynomial is real-rooted if and only if its discriminant is non-negative. This implies the lemma. \square

An immediate consequence of the lemma is that if μ is a sum of independent Bernoulli random variables then density function of μ is an ultra log concave sequence of numbers.

Figure 10.1: The roots of the hyperbolic polynomial $z_1^2 - z_2^2 - z_3^2$.

10.2 Hyperbolic and Real Stable Polynomials

We say a polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ is *homogeneous* if every monomial of p has the same degree.

Definition 10.5 (Hyperbolic Polynomials). *We say a polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ is hyperbolic with respect to a vector $e \in \mathbb{R}^n$, if p is homogeneous, $p(e) > 0$ and for any vector $x \in \mathbb{R}^n$, $p(x - te)$ is real-rooted.*

For example, the polynomial $z_1^2 - z_2^2 - \dots - z_n^2$ is hyperbolic with respect to $(1, 0, \dots, 0)$. To see that we need to show that for any $v \in \mathbb{R}^n$,

$$(x(1) - t)^2 = x(2)^2 + \dots + x(n)^2$$

is real rooted. This holds obviously because the RHS is nonnegative. Figure 10.1 shows the roots of the polynomial $z_1^2 - z_2^2 - z_3^2$ in the 3 dimensional real plane, the vertical axis shows the direction of z_1 . Observe that any vertical line has exactly two intersection with this variety, so this polynomial is hyperbolic with respect to $e = (1, 0, 0)$. As a nonexample, consider the polynomial $z_1^4 - z_2^4 - z_3^4$. This polynomial is not hyperbolic with respect to $e = (1, 0, 0)$. Geometrically similar to Figure 10.1 any vertical line crosses the variety of the roots only in two points, so this polynomial has only two real roots (as opposed to four) along each vertical line.

Hyperbolic polynomials enjoy a very surprising property, that connects in an unexpected way algebra with convex analysis. Given a hyperbolic polynomial p , it is not hard to see that the set

$$\Lambda_{++} := \{x : \text{all roots of } p(x - te) \text{ are positive}\}$$

is a convex cone. This cone is known as the hyperbolicity cone. For example, the hyperbolicity cone of the polynomial $p(z) = z_1 \dots z_n$ is the positive orthant $\{x : x(i) > 0, \forall i\}$.

As another example, for a set of variables $\{z_{i,j}\}_{1 \leq i \leq j \leq n}$, the polynomial

$$\det(Z) = \det \begin{pmatrix} z_{1,1} & \dots & z_{1,n} \\ \vdots & \ddots & \vdots \\ z_{1,n} & \dots & z_{n,n} \end{pmatrix}$$

is hyperbolic with respect to the identity matrix. This is because Z is symmetric and $\det(X - tI)$ for a symmetric matrix X is real rooted as X has real eigenvalues. In this case the hyperbolicity cone of $\det(Z)$ is the space of all symmetric PD matrices.

In general, since Λ_{++} is a convex cone and has an efficient separation oracle one can use the ellipsoid algorithm or the interior point method [Ren06; Gul97] to optimize any convex function in this domain. This is because for any point x one can test in polynomial time whether $x \in \Lambda_{++}$ by calculating the roots of the real rooted polynomial $h(x - te)$. This family of convex program are also known as the hyperbolic programming. As we have proven above, hyperbolic programming extends semidefinite programming. But it is very interesting area of research to find the extent of this generalization.

The following theorem of Gårding characterizes the hyperbolicity cone.

Theorem 10.6 (Gårding [Gar59]). *For any polynomial $p(\cdot)$ hyperbolic with respect to e , the hyperbolicity cone is the connected component of $p(x) \neq 0$ that includes e . In addition, $p(\cdot)$ is hyperbolic with respect to any direction in the hyperbolicity cone.*

Next, we study stable polynomials which are a special family of hyperbolic polynomials.

Definition 10.7 (Stable Polynomials). *For $\Omega \subseteq \mathbb{C}^d$ we say a polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$ is Ω -stable if no roots of p lies in Ω . In particular, we say p is \mathcal{H} -stable, or stable if no roots of p lies in the upper-half complex plane, i.e., $p(z) \neq 0$ for all points z in*

$$\mathcal{H}^n := \{v : \text{Im}(v) > 0, \forall 1 \leq i \leq n\}.$$

We say p is real-stable if p is \mathcal{H} -stable and all of the coefficients of p are real.

Note that a real stable polynomial is not necessarily homogeneous.

A simple example of real-stable polynomials is the polynomial

$$p(z) = \prod_{i=1}^n z_i.$$

Note that for each i , z_i is a real-stable polynomial and the product of any two real-stable polynomials is real-stable.

Suppose that $p(z)$ is a real stable polynomial; if we let $z_i = t$ for all i we get a univariate polynomial that is real-rooted (if it is not real-rooted then it must have root with positive imaginary value by Lemma 10.1, so $p(z)$ has a root in \mathcal{H}^n which is not possible). Therefore, real stability can be seen as a generalization of real-rootedness. In what follows we will see that it also has all of the closure properties of real-rootedness, so it can be seen as a natural generalization of real-rootedness.

It can be seen that a homogeneous polynomial is real-stable polynomial if it is hyperbolic w.r.t. every direction $e \in \mathbb{R}_{>0}^n$. In other words, it is real-stable if the hyperbolicity cone of the vector $-1, \dots, -1$ includes the negative orthant.

Lemma 10.8. *A polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ is real-stable if and only if for any $e \in \mathbb{R}_{>0}^n$ with positive coordinates and $x \in \mathbb{R}^n$, $p(x + te)$ is real-rooted.*

Proof. First, suppose $p(x + te)$ has an imaginary root \tilde{t} for some $x \in \mathbb{R}^n$ and $e \in \mathbb{R}_{>0}^n$. Since the complex roots of a univariate polynomial appear in conjugates Lemma 10.1, we can assume $\text{Im}(t) > 0$. But then, $z = x + \tilde{t}e$ is a root of $p(\cdot)$ and the imaginary value of each coordinate of z is positive, which is a contradiction.

Conversely, suppose for every $x \in \mathbb{R}^n$, and $e \in \mathbb{R}_{>0}^n$, $p(x + te)$ is real-rooted but $p(z)$ has a root $\tilde{z} \in \mathcal{H}^n$. Suppose for each i , $\tilde{z}_i = \tilde{z}_{i,1} + i\tilde{z}_{i,2}$. Since $\tilde{z}_i \in \mathcal{H}$, $\tilde{z}_{i,2} > 0$. Therefore, letting $x(i) = \tilde{z}_{i,1}$ and $e(i) = \tilde{z}_{i,2}$ for all i we conclude that \tilde{z} is a root of $p(x + te)$ which is a contradiction. \square

Example 10.9. A simple application of the above lemma is that for any set $\{a_1, \dots, a_n\} \in \mathbb{R}$, the polynomial $\sum_{i=1}^n a_i z_i$ is real-stable.

One of the most important family of real-stable polynomials is the determinant polynomial.

Lemma 10.10. Given PSD matrices $A_1, \dots, A_n \in \mathbb{R}^{d \times d}$ and a symmetric matrix $B \in \mathbb{R}^{d \times d}$, the polynomial

$$p(z) = \det \left(B + \sum_{i=1}^n z_i A_i \right)$$

is real stable.

Proof. By Lemma 10.8, it is enough to show that for any $x \in \mathbb{R}^n$ and $e \in \mathbb{R}_{>0}^n$,

$$p(x + te) = \det \left(B + \sum_{i=1}^n x(i) A_i + t \sum_{i=1}^n e(i) A_i \right)$$

is real-rooted. First, assume that A_1, \dots, A_n are positive definite. Then, $M = \sum_{i=1}^n e(i) A_i$ is also positive definite. So, the above polynomial is real-rooted if and only if

$$\det \left(M^{-1/2} \left(B + \sum_{i=1}^n x(i) A_i \right) M^{-1/2} + tI \right)$$

is real-rooted. The roots of the above polynomial are the eigenvalues of the matrix $M' = M^{-1/2}(B + x(1)A_1 + \dots + x(n)A_n)M^{-1/2}$. Since B, A_1, \dots, A_n are symmetric, M' is symmetric. So, its eigenvalues are real and the above polynomial is real-rooted.

If $A_1, \dots, A_n \succeq 0$, i.e., if the matrices have zero eigenvalues, then we appeal to the following theorem Hurwitz. This completes the proof of the lemma. In particular, we construct a sequence of polynomial with matrices $A_i + I/2^k$. These polynomials uniformly converge to p and each of them is real-stable by the above argument; so p is real-stable. \square

Lemma 10.11 (Hurwitz [Hur96]). Let $\{p_k\}_{k \geq 0}$ be a sequence of Ω -stable polynomials over z_1, \dots, z_n for a connected and open set $\Omega \subseteq \mathbb{C}^n$ that uniformly converge to p over compact subsets of Ω . Then, p is Ω -stable.

It also follows from the above proof that the polynomial

$$\det(z_1 A_1 + \dots + z_n A_n)$$

is hyperbolic with respect to e if $e(1)A_1 + \dots + e(n)A_n \succ 0$.

A simple application of Lemma 10.10 is that the random spanning tree polynomial is real-stable.

Corollary 10.12. For any graph $G = (V, E)$ with weights $w : E \rightarrow \mathbb{R}_+$, and $z = \{z_e\}_{e \in E}$ the following polynomial is real-stable

$$p(z) = \sum_{T \in \mathcal{T}} \prod_{e \in T} w(e) z_e.$$

Proof. By Lemma 10.10,

$$\det \left(\mathbf{1}\mathbf{1}^T + \sum_{e \in E} z_e w(e) b_e b_e^T \right)$$

is real-stable, because each matrix $w(e)b_e b_e^T$ is a PSD matrix as $w(e) \geq 0$ for all $e \in E$. By the proof of the matrix tree theorem the above polynomial is just n times $p(z)$. \square

10.2.1 Closure Properties

In general, it is not easy to directly prove that a given polynomial is real stable. Instead, one may use an indirect proof: To show that $q(z)$ is (real) stable we can start from a polynomial $p(z)$ where we can prove stability using [Lemma 10.10](#), then we apply a sequence of operators that preserve stability to $p(z)$ and we obtain $q(z)$ as the result.

In a brilliant sequence of papers Borcea and Brändén characterized the set of linear operators that preserve real stability [[BB09a](#); [BB09b](#); [BB10](#)]. We explain two instantiation of their general theorem and we use them to show that many operators that preserve real-rootedness for univariate polynomials preserve real-stability for of multivariate polynomials.

We start by showing that some natural operations preserve stability and then we highlight two theorems of Borcea and Brändén.

The following operations preserve stability.

Symmetrization If $p(z_1, z_2, \dots, z_n)$ is real stable then so is $p(z_1, z_1, z_3, \dots, z_n)$.

Specialization If $p(z_1, \dots, z_n)$ is real stable then so is $p(a, z_2, \dots, z_n)$ for any $a \in \mathbb{R}$. First, note that for any integer k , $p_k = p(a + i2^{-k}, z_2, \dots, z_n)$ is a stable polynomial (note that p_k may have complex roots). Therefore by Hurwitz theorem [10.11](#), the limit of $\{p_k\}_{k \geq 0}$ is a stable polynomial, so $p(a, z_2, \dots, z_n)$ is stable. But since all of the coefficients of $p(a, z_2, \dots, z_n)$ are real it is real stable.

External Field If $p(z_1, \dots, z_n)$ is real stable then so is $p(w_1 \cdot z_1, \dots, w_n \cdot z_n)$ for any positive vector $w \in \mathbb{R}_{>0}^n$.

Inversion If $p(z_1, \dots, z_n)$ is real stable and degree of z_i is d_i then $p(1/z_1, \dots, 1/z_n) \prod_{i=1}^n z_i^{d_i}$ is real stable.

Differentiation If $p(z_1, \dots, z_n)$ is real stable then so is $\partial p / \partial z_1$.

As an application of the above closure properties we show that for any graph G the number of edges in a (weighted) uniform spanning tree distribution is concentrated around its expected value. Recall that we proved this property by first showing the negative correlation property among any subset of edges and then proving an extension of the Chernoff bound.

Corollary 10.13. *For any graph $G = (V, E)$ and weights $w : E \rightarrow \mathbb{R}_+$, let μ be a weighted uniform spanning tree distribution in G . Then, for any set $F \subseteq E$,*

$$\mathbb{P}_{T \sim \mu} [|T \cap F| > (1 + \delta) \mathbb{E} [|T \cap F|]] \leq \left(\frac{e^\delta}{(1 + \delta)^{1 + \delta}} \right)^{\mathbb{E} [|T \cap F|]}.$$

Proof. First, by [Corollary 10.12](#),

$$p(z) = \sum_{T \in \mathcal{T}} \prod_{e \in E} w(e) z_e,$$

is real stable. Let $p_1(z)$ be a specialization of $p(z)$ where for each $e \notin F$, we let $z_e = 1$. In words, $p_1(z)$ can be seen as projecting the spanning tree distribution to the set F . So, p_1 is real stable. Now, let $p_2(t)$ be a univariate polynomial where we set all variables of $p_1(z)$ equal to t , so p_2 is real stable, i.e., it is real rooted. Let $a_0, \dots, a_{|F|}$ be the coefficients of $p_2(t)$. Since p_2 is real rooted there is a set of $|F|$ independent Bernoulli random variables where the law of their sum is exactly $a_0, \dots, a_{|F|}$ (and the sum of their expectation is $\mathbb{E} [|T \cap F|]$). But, by Chernoff bound, the sum of any set of Bernoulli random variables is concentrated around its expectation; the statement follows. \square

In fact, the above corollary shows that a (weighted) uniform distribution of random spanning trees is not very far from an independent distribution, i.e., if we symmetrize the edges we see a distribution identical to an independent distribution. Using Newton identities we can also show that the distribution of the random variable $|T \cap F|$ in the above corollary is (ultra) log concave and unimodal. This is a remarkable consequence of the real stability theory; we are not aware of any such proof that only uses the negative correlation property.

Next we describe two instantiations of the Borcea and Brändén operator preserving theorems.

Let $\mathcal{A}_n[\mathbb{R}]$ be the set all finite order linear differential operators with real coefficients. For vectors $\alpha, \beta \in \mathbb{N}^n$, we write

$$z^\alpha = z_1^{\alpha(1)} \dots z_n^{\alpha(n)}, \partial^\alpha = \frac{\partial^{\alpha(1)}}{\partial z_1^{\alpha(1)}} \dots \frac{\partial^{\alpha(n)}}{\partial z_n^{\alpha(n)}}.$$

Then, each operator $D \in \mathcal{A}_n[\mathbb{R}]$ can be uniquely represented as

$$D = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha, \beta} z^\alpha \partial^\beta.$$

A nonzero differential operator $D \in \mathcal{A}[\mathbb{R}]$ is called *stability preserver* if it maps any real stable polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ to another real stable polynomial $Dp \in \mathbb{R}[z_1, \dots, z_n]$.

Theorem 10.14. *Let $D \in \mathcal{A}_n[\mathbb{R}]$. Then, D is stability preserver if and only if*

$$\sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha, \beta} z^\alpha (-w)^\beta \in \mathbb{R}[z_1, \dots, z_n, w_1, \dots, w_n]$$

is a real stable polynomial.

Let us give some simple application of the above theorem. By [Example 10.9](#), if $p(z)$ is real stable then so is $(1 - \partial_{z_1})p(z)$, or in fact any polynomial of the form

$$\sum_{i=1}^n a_i \partial_{z_i} p(z).$$

As a nonexample, consider the operator $D = (1 - \partial_{z_1}^3)$. Observe that the polynomial $1 - (-w)^3$ is not real stable, for $w = e^{2i\pi/3}$. In fact if $p(z) = z_1^3$, then

$$(1 - \partial_{z_1}^3)p(z) = z_1^3 - 6,$$

is not a stable polynomial.

As another application of the above theorem we can show that the degree 2 symmetric polynomial, $z_1 z_2 + z_2 z_3 + z_1 z_3$ is real stable. Recall that $z_1 z_2 z_3$ is real stable as it is a product of real stable polynomials. Therefore,

$$\sum_{i=1}^3 \frac{\partial}{\partial z_i} z_1 z_2 z_3 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

is real stable. We leave it as an exercise to show that the degree k elementary symmetric polynomial in n variables for any $n > k$ is real stable.

Next, we discuss a closure property of real stable polynomial when we scale the coefficients of the polynomial by a *multiplier sequence*.

Theorem 10.15 (Pólya-Schur). *For a sequence of real numbers w_0, w_1, \dots , the linear operator T_w is defined as follows:*

$$T_w \left(\sum_{i=0}^n a_i z^i \right) = \sum_{i=0}^n w_i a_i z^i.$$

We say w is a multiplier sequence if T_w preserve the stability (real rootedness) of the given stable polynomial. Then, w is a multiplier sequence if and only if $T_w((1+z)^n)$ is real stable with all roots of the same sign.

Borcea and Brändén extend the above theorem to multivariate stable polynomials. Think of $w = w : r \in \mathbb{Z}_+^d \rightarrow \mathbb{R}$ as a function of real number scalars; in particular, for any integer vector $r \in \mathbb{Z}_+^d$, $w(r)$ is the scalar of the monomial z_r where the degree of z_i is $r(i)$. We say w is a multiplier sequence if

$$T_w \left(\sum_r a_r z^r \right) = \sum_r w(r) a_r z^r$$

preserves the class of real stable polynomials.

Theorem 10.16 ([BB10, Theorem 1.8]). *The array w is a d -variate multiplier sequence if and only if there are d univariate multiplier sequences w_1, \dots, w_d such that for all r ,*

$$w(r) = w_1(r(1)) \dots w_d(r(d))$$

and satisfying a further sign condition: Either every $w(r)$ is nonnegative or every $w(r)$ is nonpositive or the same holds for $(-1)^{|r|} w_r$.

The following corollary is immediate

Corollary 10.17. *Let p be a real stable polynomial, then the multi-affine part of p , i.e., the sum of all square free monomials of p is real stable.*

Proof. For every $1 \leq i \leq d$, let $w_i(0) = w_i(1) = 1$ and $w_i(j) = 0$ for all $j \geq 2$. The corollary follows from [Theorem 10.16](#). \square

Let us give an application of the corollary. Say we have two variables z_e for each edge e in a graph G . As we discussed in [Corollary 10.13](#), $p(z) = \sum_{T \in \mathcal{T}} \prod_{e \in T} w(e) z_e$ is real stable. Since the product of any two real stable polynomials is real stable, $p(z)^2$ is real stable. Now, by the above corollary, the sum of square free monomials of $p(z)^2$ is real stable, so

$$\sum_{\substack{F \text{ is a union of} \\ \text{two disjoint trees}}} \prod_{e \in F} z_e n(F)$$

is real stable, where $n(F)$ is the number of partitioning of F into two disjoint spanning trees.

10.3 Characterization of Real Stability

Helton and Vinnokov [\[HV07\]](#) characterized the set of Hyperbolic polynomials in three variables.

Theorem 10.18 ([HV07; LPR05]). *Suppose that $p(x, y, z)$ is of degree d and hyperbolic with respect to $e \in \mathbb{R}^3$. Suppose further that p is normalized such that $p(e) = 1$. Then there are symmetric matrices $A, B, C \in \mathbb{R}^{d \times d}$ such that*

$$p(x, y, z) = \det(xA + yB + zC),$$

where $e(1)A + e(2)B + e(3)C = I$.

This result was a major breakthrough in this area, it made many of the classical proofs simpler and it has been used since then quite often.

Borcea and Brändén [BB10] extended the above theorem to the family of real stable polynomials. Roughly speaking, if p is real stable, that is if p is hyperbolic with respect to any $e \in \mathbb{R}_{>0}^2$, then $p(\cdot)$ can be written as the determinant of $xA + yB + C$ subject to that $A, B \succeq 0$

Theorem 10.19 (Borcea and Brändén [BB10]). *If $p(x, y) \in \mathbb{R}[x, y]$ of degree d is real stable, then there exist PSD matrices $A, B \in \mathbb{R}^{d \times d}$ and a symmetric matrix $C \in \mathbb{R}^{d \times d}$ such that*

$$p(x, y) = \det(xA + yB + C).$$

The above theorem shows that the converse of Lemma 10.10 is true if the polynomial has at most two variables. If $p(x, y, z)$ is a homogeneous real stable polynomial then we can also write it in terms of determinants,

$$p(x, y, z) = \det(xA + yB + zC)$$

for $A, B, C \succeq 0$. In the above theorem we restrict to two-variate polynomials because $p(\cdot)$ may not be homogeneous.

Let us give some examples and non-examples of the above results. As we discussed above, the polynomial $z_1^2 - z_2^2 - z_3^2$ is hyperbolic with respect to $(1, 0, 0)$. We can write

$$\begin{aligned} z_1^2 - z_2^2 - z_3^2 &= \det \begin{bmatrix} z_1 + z_2 & z_3 \\ z_3 & z_1 - z_2 \end{bmatrix} \\ &= \det \left(z_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + z_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right), \end{aligned}$$

and the coefficient of z_1 is I .

As another example, recall that $z_1z_2 + z_2z_3 + z_3z_1$ is a homogeneous real stable polynomial. So, by the above theorem it can be written as a determinant polynomial.

$$\begin{aligned} z_1z_2 + z_2z_3 + z_1z_3 &= \det \begin{bmatrix} z_1 + z_3 & z_3 \\ z_3 & z_2 + z_3 \end{bmatrix} \\ &= \det \left(z_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + z_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right). \end{aligned}$$

As a non-example the real stable polynomial $\sum_{1 \leq i < j \leq 4} z_i z_j$ can not be written as

$$\det \left(\sum_{i=1}^4 z_i A_i \right)$$

where $A_i \in \mathbb{R}^{2 \times 2}$ is a real symmetric matrix. Roughly speaking, the reason is that the space of 2 by 2 real symmetric matrices is 3 dimensional and we have more than 3 variables.

It was conjectured that if $p(\cdot)$ is hyperbolic then at least a higher power of $p(\cdot)$ can be written as a determinant. This conjecture makes sense because higher powers of $p(\cdot)$ are supported on the same set of variables but have much higher degree so the dimension of the matrices A_i can be significantly higher than the number of variables. This conjecture is recently refuted by Brändén [Bra11].

10.4 Interlacing Polynomials

Definition 10.20 (Interlacing). We say a real rooted polynomial $p(t) = \alpha_0 \prod_{i=1}^{d-1} (t - \alpha_i)$ interlaces $q(t) = \beta_0 \prod_{i=1}^d (t - \beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \cdots \leq \alpha_{d-1} \leq \beta_d.$$

We also say p, q are interlacing if the degree of p, q differ by at most 1 and their roots alternate.

As a simple example, for any univariate real rooted polynomial $p(z)$, the polynomial dp/dz interlaces $p(z)$.

By convention, we assume that the roots of any two polynomials of degree 0 or 1 interlace. The *Wronskian* of two polynomials p, q is defined as follows:

$$W[p, q] = p'q - pq'. \quad (10.1)$$

It is not hard to show that if p, q are interlacing, then $W[p, q]$ is either nonnegative or nonpositive in the whole real axis. In addition, if p interlaces q , then $W[p, q] \leq 0$ on the whole real axis.

Lemma 10.21. Let $p(t) = \alpha_0 \prod_{i=1}^m (t - \alpha_i)$ and $q(t) = \beta_0 \prod_{i=1}^n (t - \beta_i)$ where $|n - m| \leq 1$. If the roots of p, q alternate, then either $W[p, q] \leq 0$ or $W[p, q] \geq 0$ on the whole real axis.

Proof. First, we show that without loss of generality we can assume p, q do not have any common roots. If α is a common root, then we inductively prove that for $p_1(t) = p(t)/(t - \alpha)$ and $q_1(t) = q(t)/(t - \alpha)$, the sign of $W[p_1, q_1]$ is invariant over \mathbb{R} , so the sign of

$$\begin{aligned} W[p, q] &= q(t)(p_1(t) + (t - \alpha)p_1'(t)) - p_1(t)(q_1(t) + (t - \alpha)q_1'(t)) \\ &= (t - \alpha)^2(p_1'(t)q_1(t) - p_1(t)q_1'(t)), \end{aligned}$$

is also invariant as $(t - \alpha)^2$ is non-negative.

So, suppose p, q do not have any common roots. Since, p, q are interlacing, the multiplicity of each root of p and each root of q is 1. We show a stronger claim, that is $W[p, q] \neq 0$ over the whole real axis, this proves the lemma because of continuity of the Wronskian. Suppose $W[p, q] = 0$ at a point t^* . Therefore,

$$p'(t^*)q(t^*) = p(t^*)q'(t^*).$$

First assume t^* is a root $p(t^*) = 0$. But then $q(t^*) \neq 0$ because p, q do not have common roots and $p'(t^*) \neq 0$ because the multiplicity of roots of p (and q) are 1. But, this is a contradiction.

Now, suppose $p(t^*) \neq 0$ and $q(t^*) \neq 0$. Then,

$$\sum_{i=1}^m \frac{1}{t^* - \alpha_i} = \frac{p'(t^*)}{p(t^*)} = \frac{q'(t^*)}{q(t^*)} = \sum_{i=1}^n \frac{1}{t^* - \beta_i}.$$

Without loss of generality assume α_k is the largest root that is less than t^* and β_{k+1} is the smallest root larger than t^* . We show the LHS is strictly bigger than the right. This is because for each $i \leq k$,

$$\frac{1}{t^* - \alpha_i} > \frac{1}{t^* - \beta_i} > 0.$$

In addition, for each $i > k$,

$$\frac{1}{t^* - \beta_i} < \frac{1}{t^* - \alpha_i} < 0.$$

But, this is a contradiction, so the Wronskian is nonzero in the whole \mathbb{R} . □

We note that the converse of the above lemma also holds if the multiplicity of the roots of p, q is 1. Otherwise, the Wronskian may not change sign while p, q do not interlace.

Example 10.22. Let $p(t) = t^3(t-2)$ and $q(t) = (t-1)^3(t+1)$. Then, p, q have the same degree but they do not interlace. However, the Wronskian $W[p, q] \geq 0$ over all \mathbb{R} ,

$$p'q - q'p = 2(t-1)^3(t+1)t^2(2t-3) - 2(t-2)(t-1)^2t^3(2t+1) \geq 0.$$

If $W[p, q] \leq 0$, then we say p, q are in *proper position* and we write $p \ll q$.

Next, we highlight several theorems about necessary and sufficient conditions for interlacing.

Theorem 10.23 (Hermite-Biehler). *For any two polynomials $p, q \in \mathbb{R}[z]$, $p + iq$ is real rooted if and only if $q \ll p$.*

Theorem 10.24 (Obrechhoff, Dedieu [Ded92]). *Any two real-rooted degree d polynomials p, q are interlacing if and only if $ap + bq$ is real rooted for any $a, b \in \mathbb{R}$.*

If our assumption is that any convex combination of p, q is real rooted it is not necessarily true that p, q are interlacing but we can show that they have a common interlacer.

Theorem 10.25 (Dedieu [Ded92]). *For any two real-rooted degree d polynomials p, q , with roots $\alpha_1, \dots, \alpha_d$ and β_1, \dots, β_d , the following are equivalent:*

- i) *For any $a \in [0, 1]$, $ap + (1-a)q$ is real-rooted.*
- ii) *p, q have a common interlacer, i.e., for any $1 \leq i < d$,*

$$\max(\alpha_i, \beta_i) \leq \min(\alpha_{i+1}, \beta_{i+1}).$$

Hermite-Biehler theorem gives a good indication of a generalization of the concept of interlacing to multivariate polynomials.

Definition 10.26. *Two multivariate polynomials $p, q \in \mathbb{R}[z_1, \dots, z_n]$ are in proper position, $p \ll q$ if for all $x \in \mathbb{R}^n$ and $e \in \mathbb{R}_{>0}^n$,*

$$p(x + te) \ll q(x + te).$$

Roughly speaking, the above definition says that two multivariate polynomials interlace if along every positive direction $e \in \mathbb{R}_{>0}^n$ they interlace. Note that for univariate polynomials the two definitions coincide. The following theorem is proved in [BBS09]

Theorem 10.27 (Borcea, Brändén, Shapiro [BBS09]). *For any $p, q \in \mathbb{R}[z_1, \dots, z_n]$, $p + z_{n+1}q$ is real stable if and only if $q \ll p$.*

The above theorem relates real stable polynomials and interlacing. Say $p + z_{n+1}q$ is the real stable polynomial corresponding to the random spanning tree distribution of a graph G with $n+1$ edges (see [Corollary 10.12](#)). Let e_{n+1} be the edge corresponding to z_{n+1} . In this case p is the polynomial of uniform spanning tree distribution on $G \setminus e_{n+1}$ and q is the polynomial of uniform spanning tree distribution on G/e_{n+1} , i.e., when e_{n+1} is contracted. The above theorem says that these two polynomials interlace along every positive direction in the space.

Let us give an illuminating example. Let G be a cycle of length 3 with edges e_1, e_2, e_3 . This graph has 3 spanning trees, so the corresponding polynomial of a uniform distribution on this trees is

$$z_1z_2 + z_2z_3 + z_1z_3 = z_1z_2 + z_3(z_1 + z_2)$$

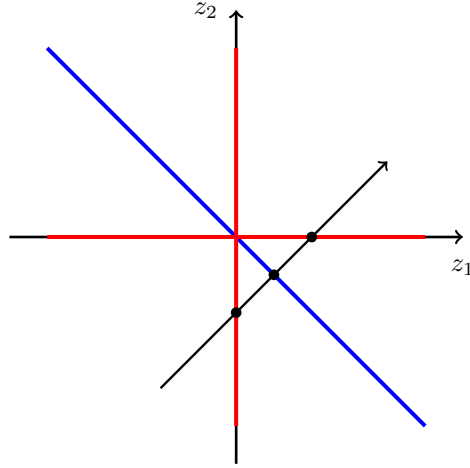


Figure 10.2: The red lines show the zeros of the polynomial $z_1 \cdot z_2$ in the real plane, the blue line shows the zeros of the polynomial $z_1 + z_2$. Observe that any line pointing to the positive orthant crosses the blue line once in the middle of the red lines, so the roots interlace.

Following the argument in the previous paragraph, we have $p = z_1 \cdot z_2$ and $q = z_1 + z_2$. In [Figure 10.2](#) we plotted the roots of these two polynomials. Observe that the roots of the polynomials interlace along any line pointing to the positive orthant, so $q \ll p$.

Next we show that stability of real stable polynomials is closely related to negative correlation. The following

Theorem 10.28 (Brändén [[Bra07](#), Theorem 5.6]). *A multilinear polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ is real stable if and only if for all $1 \leq i < j \leq n$ and $x \in \mathbb{R}^n$,*

$$\frac{\partial p}{\partial z_i}(x) \cdot \frac{\partial p}{\partial z_j}(x) - p(x) \cdot \frac{\partial p}{\partial z_i \partial z_j}(x) \geq 0$$

Note that the multilinearity is a necessary condition for the above theorem. That is, if a real stable polynomial is not multilinear, then the above inequality may not hold for all $x \in \mathbb{R}^n$.

Example 10.29. Let $p(z_1, z_2) = z_1^2 + z_2^2$. It is easy to see that p is a real stable polynomial but it is not multilinear. Now, for $i = 1, j = 2$ the above inequality is equivalent to

$$4z_1z_2 - (z_1^2 + z_2^2) \geq 0$$

The above inequality does not hold for the point $z_1 = 1, z_2 = -1$.

Let us give some interesting applications of the above theorem. Let p be the polynomial of random spanning trees of a graph G and let z_i and z_j be the variables corresponding to e_i, e_j . Also, let $x_i = 1$ for all i . Then, $p(x) = 1$, $\frac{\partial p}{\partial z_i}(x) = \mathbb{P}[e_i \in T]$, and $\frac{\partial p}{\partial z_i \partial z_j}(x) = \mathbb{P}[e_i, e_j \in T]$. So, when $x = 1$ the above inequality implies that e_i, e_j are negatively correlated. Roughly speaking, one can read the above inequality as that the random spanning tree distribution is negatively correlated along every direction.

Although the above statement does not make any sense when we look at a probability distribution, we can interpret by looking at the polynomial of a distribution (see [Definition 10.33](#) for the polynomial of a distribution). Let $\partial_i p = \frac{\partial p}{\partial z_i}$. Also, let $p = f + z_i h$ for $f, h \in \mathbb{R}[z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n]$. Then,

$$\begin{aligned} \partial_i p \partial_j p - p \cdot \partial_{i,j} p &= h \cdot (\partial_j f + z_i \partial_j h) - \partial_j h \cdot (f + z_i h) \\ &= h \partial_j f - f \partial_j h. \end{aligned}$$

By [Theorem 10.27](#) and that p is real stable we know that f, h are interlacing along any line pointing to the positive orthant. So $W[h, f] \leq 0$ along the direction that is 1 in the j -th coordinate and 0 everywhere else. Therefore, the quantity in the RHS of the above equation is always nonnegative.

The Upshot. The above argument shows that interlacing is the same as negative correlation when we look at multilinear polynomials. Intuitively, interlacing along every direction is a generalization of negative correlation property which implies real stability ([Theorem 10.27](#)). Real stability implies strongest form of negative dependence properties (see [section 10.6](#)).

10.5 Negative Dependence

We discuss negative correlation and one of its very important consequences, i.e., concentration of measures, in lectures 2 and 3. Negative correlation (respectively positive correlation) can be seen as the weakest form negative dependent between a given set (binary) random variables. Roughly speaking, when a distribution is negative correlated we can argue that the moments of the distribution are at most what they would be in the independent case. This property implies strong concentration bounds.

An stronger form of dependence is called negative (respectively positive) association. We say a collection of binary random variables X_1, \dots, X_n are positively associated if for any two nondecreasing function $f, g : \{0, 1\}^n \rightarrow \mathbb{R}$,

$$\mathbb{E}[f(X_1, \dots, X_n)] \cdot \mathbb{E}[g(X_1, \dots, X_n)] \leq \mathbb{E}[f(X_1, \dots, X_n)g(X_1, \dots, X_n)]. \quad (10.2)$$

Recall that a function $f : \{0, 1\}^n$ is nondecreasing if for any two vectors $X_1, \dots, X_n, Y_1, \dots, Y_n \in \{0, 1\}^n$ where $X_i \leq Y_i$ for all i ,

$$f(X_1, \dots, X_n) \leq f(Y_1, \dots, Y_n).$$

We also say f is nonincreasing if $-f$ is nondecreasing.

Negative association cannot be defined by simply reversing the inequality (10.2). This is because any function is always nonnegatively correlated with itself. Therefore, we can only hope to get negative correlation when f, g depend on disjoint set of variables. In particular, we say $\{X_1, \dots, X_n\}$ are negatively associated if for any two nondecreasing functions f, g such that for a set $S \subset [n]$, f only depends on $\{X_j : j \in S\}$ and g depends only on the rest of the variables,

$$\mathbb{E}[f(X_1, \dots, X_n)] \cdot \mathbb{E}[g(X_1, \dots, X_n)] \geq \mathbb{E}[f(X_1, \dots, X_n)g(X_1, \dots, X_n)]. \quad (10.3)$$

One of the most useful results on positive association is a result of Fortuin, Kasteleyn and Ginibre [[FKG71](#)]. Consider a probability distribution μ on $\{0, 1\}^n$, we say μ has the positive lattice condition if for any two vectors $x, y \in \{0, 1\}^n$,

$$\mu(x \wedge y) \cdot \mu(x \vee y) \geq \mu(x) \cdot \mu(y). \quad (10.4)$$

Theorem 10.30. *If μ satisfies the positive lattice condition, then μ is positively associated.*

Let us give a motivating example. Suppose we have Erdős-Rényi random graph $G(n, p)$ for some $0 < p < 1$. We can see the edge set of this graph as a sample in $\{0, 1\}^{\binom{n}{2}}$. Because of the edges are sampled independent for any point $x \in \{0, 1\}^{\binom{n}{2}}$, $\mu(x) = p^{\|x\|_1} \cdot (1-p)^{\binom{n}{2}-\|x\|_1}$, where we use $\|x\|_1$ to denote the number of ones of x . It follows that this distribution satisfies (10.4) with equality for all x, y . Therefore, it is positively associated (note that because of the independence property this distribution is also negatively associated). Consequently any two increasing functions are positively associated. For example, say f is the function

indicating that G has a Hamiltonian cycle and g is the function indicating that G is 3-colorable. Observe that f is a nondecreasing function and g is a nonincreasing function. So, by (10.2), $f, -g$ are positively associated or f, g are negatively associated. Note that the striking fact is that f, g are NP-hard functions but we can study their correlation thorough the lens of positive association.

Although we have such a strong theorem for positive association, the negative dependence and negative association were not fully understood until very recently. We say a matroid is *balanced* if the uniform distribution of the bases of the matroid and the bases of all of its minors are negatively correlated. Feder and Mihail [FM92] used the following theorem to prove that balanced matroids are negatively associated. Recall that a probability distribution μ is homogeneous if every set in the support of μ has the same size.

Theorem 10.31 (Feder and Mihail [FM92]). *Given a class of homogeneous measures on finite Boolean algebras (of differing sizes) which is closed under conditioning. If each of the measures in this class satisfy pairwise negative correlation, then all measures in this class are negatively associated.*

The above theorem implies that random spanning trees and more generally linear matroids are negatively associated probability distributions.

Negative association is only one form of negative dependence. The following fact is a simple application of the negative association property of random spanning trees.

Fact 10.32. *For any graph $G = (V, E)$, any (weighted) uniform spanning tree distribution μ , $F \subseteq E$, $c \in \mathbb{R}$ and any edge $e \notin F$,*

$$\mathbb{P}_{T \sim \mu} [e \in T \mid |S \cap T| \geq c] \leq \mathbb{P}_{T \sim \mu} [e \in T].$$

Pemantle [Pem00] wrote a nice survey and asked for general theories regarding properties of negatively dependent probability distributions. Several years later, Borcea, Brändén and Liggett used the rich theory of real stable polynomials to answer many of the questions raised in [Pem00]. They introduced Strongly Rayleigh measures as a family of probability distributions whose *generating polynomial* is a real stable polynomial and they show that this family satisfy almost all of the properties that Pemantle asked in his survey.

10.6 Strongly Rayleigh Distributions

In this section we introduce strongly Rayleigh measures and their properties. The material of this section are mostly based on the seminal work of Borcea, Brändén and Liggett [BBL09]. In many cases we will discuss interesting consequences of these properties in the random spanning tree distributions. Throughout this section we use $[n] = \{1, 2, \dots, n\}$.

Definition 10.33 (Strongly Rayleigh distributions). *For a set E of elements let $\mu : 2^{[n]} \rightarrow \mathbb{R}_+$ be a probability distribution. For a set of variables $\{z_1, \dots, z_n\}$, The generating polynomial of μ , g_μ is defined as follows:*

$$g_\mu(z) := \sum_{S \subseteq [n]} \prod_{j \in S} z_j.$$

We say μ is strongly Rayleigh (SR) if g_μ is a real stable polynomial.

Note that by the above definition, the generating polynomial of any strongly Rayleigh measure is multilinear. So, we can use Theorem 10.28 to show that any strongly Rayleigh measure is negatively correlated, and in fact more generally they are negatively correlated along every direction pointing to positive orthant.

It is an interesting question to extend the above definition to a family of distribution defined on random variables that take values in larger field, say $\{0, \dots, q\}$.

It follows by [Corollary 10.12](#) that (weighted) uniform spanning tree distributions are strongly Rayleigh. In general, it is not hard to see that any determinantal probability distribution is also SR (see Lecture 2 for the definition of determinantal probability distribution). Other examples, includes product distributions and exclusion process measures. As we will many other families of probability distributions are SR because their generating polynomial can be derived from the generating polynomial of say a random spanning tree distribution using the closure properties of real stable polynomials.

In the rest of this section we discuss closure properties and negative dependence properties of SR measures. In the next lecture we use these properties of SR to design an approximation algorithm for *symmetric* traveling salesman problem.

10.6.1 Closure Properties of SR Measures

We start by going over the closure properties of the strongly Rayleigh measures. Unless otherwise specified, throughout this section we assume $\mu : 2^{[n]} \rightarrow \mathbb{R}_+$ is a SR distribution and for each $1 \leq j \leq n$ we use X_j to denote the indicator random variable of j being in a sample of μ . Also, for a set $S \subseteq [n]$ we use $X_S := \sum_{j \in S} X_j$.

Conditioning. For any $1 \leq j \leq n$, $\{\mu|X_j = 0\}$ and $\{\mu|X_j = 1\}$ are SR. To see this observe that (up to normalization)

$$g_{\{\mu|X_j=0\}}(z) = g_\mu(z_1, \dots, z_{j-1}, 0, z_{j+1}, \dots, z_n),$$

and recall that by the specialization property of real stable polynomials the RHS is real stable. On the other hand,

$$g_{\{\mu|X_j=1\}}(z) = z_j \partial_j g_\mu(z),$$

and recall that real stable polynomials are closed under differentiation. The above property may not be too striking as random spanning tree distributions are also closed under conditioning.

Projection. For any set $S \subseteq [n]$, the projection of μ onto S , $\mu|_S$ is the measure μ' where for any $A \subseteq S$,

$$\mu'(A) = \sum_{B \subseteq [n]: B \cap S = A} \mu(B).$$

It is easy to see that $\mu|_S$ is SR. This is because for any $S \subseteq [n]$, we can construct $g_{\mu|_S}(z)$ simply by specializing $z_j = 1$ for all variables $j \notin S$. This is a very nice property, as random spanning tree distributions are not closed under projection, e.g., the distribution of the set of edges of a random spanning tree in a cut (S, \bar{S}) is not a random spanning tree distribution.

As a practical application of this property, take another look at the proof of [Corollary 10.13](#) where we used the projection property to prove concentration inequalities of random spanning tree distributions.

Truncation. For any pair of integers $1 \leq k, \ell \leq n$, the truncation of μ to $[k, \ell]$ is the conditional measure $\mu_{k,\ell}$ where

$$\mu_{k,\ell}(S) \propto \begin{cases} \mu(S) & \text{if } k \leq |S| \leq \ell \\ 0 & \text{otherwise.} \end{cases}$$

Borcea et al. show that if $|\ell - k| \leq 1$ then $\mu_{k,\ell}$ is SR. The proof of this property essentially follows from [Theorem 10.16](#). First, note that if μ is homogeneous, then truncation is a trivial operation. But, if μ is not homogeneous, truncation is a very nontrivial operation. To prove this statement [\[BBL09\]](#) first show that one can homogenize a real stable polynomial $p(\cdot)$ by adding a new dummy variable as long as all of the coefficients of $p(\cdot)$ are nonnegative, i.e., say p has degree d , they add a new variable z_{n+1} and for all $0 \leq j \leq d$ they multiply any monomial of p of degree j with z_{n+1}^{d-j} . Then, they [Theorem 10.16](#) to extract monomials of the new polynomial where the power of z_{n+1} is in $[k, \ell]$.

For example, suppose for a set of independent Bernoulli random variables B_1, \dots, B_n , $\mu(S)$ is the probability that $B_j = 1$ for all $j \in S$ and zero otherwise. Then, μ is not a homogeneous distribution. By above argument, the truncation of μ to k , μ_k , is a SR distribution; this is a nontrivial property because unlike μ , μ_k is not an independent probability distribution. As we will see later, because μ_k is SR we can argue that it is a negatively associated probability distribution.

10.6.2 Negative Dependent Properties of SR Measures

In this part we explain the negative dependent properties of SR measures.

Log Concavity of the Rank Sequence. The rank sequence of μ is a sequence q_0, q_1, \dots, q_n where $q_i = \mathbb{P}_{S \sim \mu}[S = i]$. It follows that the rank sequence of any SR measure is (ultra) log concave. The proof of this is very similar to [Corollary 10.13](#). We symmetrize all variables of g_μ ,

$$p(t) = g_\mu(t, t, \dots, t)$$

by symmetrization property, $p(t)$ is real stable. But, $p(t)$ is just a sum of independent Bernoulli variables. So, by Newton identities we get the log concavity.

Negative Association. In the above we already argued that any SR is negatively correlated. But, indeed any SR measure is negatively associated. This property follows from [Theorem 10.31](#). For simplicity suppose μ is homogeneous. Then, we can construct a class of homogeneous negatively correlated probability distributions because SR measures are closed under conditioning and all of them are negatively correlated. Then, [Theorem 10.31](#) implies that μ and any other measure in this class is negatively associated.

Stochastic Dominance Property. We say a probability event $\mathcal{A} \subseteq 2^{[n]}$ is upward closed if for any $S \in \mathcal{A}$ all supersets of S are also in \mathcal{A} . We say a probability distribution μ stochastically dominate ν , $\nu \preceq \mu$, if for any upward closed probability event \mathcal{A} , $\nu(\mathcal{A}) \leq \mu(\mathcal{A})$. Borcea et al. showed that for any integer $k < n$ if μ_k and μ_{k+1} are nonzero, then

$$\mu_k \preceq \mu_{k+1}.$$

We will not talk about the proof of this property and we refer the interested readers to [\[BBL09; Pem13\]](#). For example, if μ_k, μ_{k+1} are well defined for any $j \in F$,

$$\mathbb{E}_{\mu'_k}[X_j] \leq \mathbb{E}_{\mu'_{k+1}}[X_j]$$

The stochastic dominance property is a natural property of SR measures, roughly speaking, it says that truncation on larger numbers increases the probability of the underlying elements and upward event defined on them.

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