**Recent Advances in Approximation Algorithms** **Spring 2015**

# Lecture 9: Pipage Rounding Method

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**Disclaimer**: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In this lecture we study a different randomized rounding method called *pipage rounding method*. The materials of this lecture are based on the work of Chekuri, Vondrák and Zenklusen [CVZ10].

This method is stronger than the maximum entropy rounding by sampling method in some aspects and is weaker in some other aspects. From a highlevel point of view, one can use this method to round a fractional point in a matroid polytope to a basis in polynomial time making sure that the underlying elements of the matroid are negatively correlated. So, in particular, one can use this method to round a fractional point in the spanning tree polytope to a thin spanning tree similar to the idea in the last lecture.

But as we will see in the next few lectures, the pipage rounding method does not necessarily satisfy several of the strong negative dependence properties that are satisfied by the maximum entropy distributions of spanning trees.

## 9.1 Matroids

We start this lecture by giving a short overview of matroids.

**Definition 9.1.** *For a set $E$ of elements and $\mathcal{I} \subseteq w^E$, we say $\mathcal{M}(E, \mathcal{I})$ is a matroid, if*

1. *For all $S \in \mathcal{I}$ and $T \subseteq S$, $T \in \mathcal{I}$. This is also known as the downward closed property.*
2. *For all $S, T \in \mathcal{I}$ such that $|S| < |T|$ there is an element $e \in T \setminus S$ such that $S \cup \{e\} \in \mathcal{I}$.*

*We say $S$ is an independent set of $\mathcal{M}$ if $S \in \mathcal{I}$. We also say $T$ is a base of $\mathcal{M}$ if $T \in \mathcal{I}$ and $\mathcal{I}$ has no independent set of size larger than $|T|$.*

As an example, it is easy to see that spanning trees of a graph form the bases of a matroid. Let $\mathcal{M} = (E, \mathcal{F})$ where $E$ is the set of edges of $G$ and $\mathcal{F}$ is the set of forests of $G$, i.e., for any $S \in \mathcal{F}$ is a set of edges with not cycles. Obviously, $\mathcal{F}$ is downward closed. In addition, if $|T| > |S|$ and $S, T \in \mathcal{F}$, then $T$ has an edge that connects two connected components of $S$. So, we can add that edge to $S$ making sure that it remains a forest. It follows that $\mathcal{M}$ is a matroid, a.k.a., the graphic matroid and its bases are the spanning trees of $G$.

The rank function of a matroid is a function $r : 2^E \to \mathbb{N}$ such that for any $S \subseteq E$, $r(S)$ is the size of the largest independent set of $\mathcal{M}$ that is contained in $S$,

$$r(S) = \max\{|T| : T \in \mathcal{I}, T \subseteq S\}.$$

One of the important properties of the rank function is that it is a submodular function, i.e.,

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T), \forall S, T.$$

We will see applications of this property throughout the lecture.

A famous example of matroids is the linear matroid where $E$ corresponds to a set of vectors in a vector space and a $\mathcal{I}$ is the sets of vectors that are linearly independent. Many examples of matroids are just special cases of the linear matroid, e.g., the graphic matroid.

### 9.1.1 Matroid Polytope

Given a matroid $\mathcal{M}$, the matroid polytope is the convex hull of all vectors corresponding to the independent sets of $\mathcal{M}$. For a set $S \subseteq E$ and $x : E \to \mathbb{R}$, let $x(S) := \sum_{e \in S} x(e)$.

$$\begin{aligned} x(S) &\leq r(S) && \forall S \subseteq E, \\ x(e) &\geq 0 && \forall e \in E. \end{aligned}$$

It is easy to see that the above polytope contains all independent sets of $\mathcal{M}$. In addition its vertices are just the independent sets.

We can also characterize the matroid base polytope, that is the convex hull of all indicator vectors of the bases of $\mathcal{M}$.

$$\begin{aligned} x(E) &= r(E) && \\ x(S) &\leq r(S) && \forall S \subseteq E, \\ x(e) &\geq 0 && \forall e \in E. \end{aligned}$$

For example, observe that the spanning tree polytope that we described in the last lecture is the same as the above polytope when $\mathcal{M}$ is the graphic matroid of $G$.

Although the above polytopes have exponentially many constraints one can use the ellipsoid algorithm to test if a given point $x$ is in the matroid (base) polytope. To see that one can use some general theorems that say separation and optimization are equivalent algorithmic tasks, optimizing a linear function over a matroid is easy so is the separation problem. Another way to see this is that for a given point $x$ it is enough to test if for any set $S$, $x(S) \leq r(S)$. We define a function $f : 2^E \to \mathbb{R}$ where $f(S) = r(S) - x(S)$. It is easy to see that $f$ is a submodular function. So all we need to do is to find the minimum of a submodular function. This is a very well studied problem and it can be done in polynomial time for any given submodular function [GLS81].

## 9.2 Pipage Rounding Method

In this section we describe the pipage rounding algorithm. This algorithm is first proposed by Ageev and Sviridenco [AS04] to round a fractional matching into an integral one. Chekuri, Vondrák and Zenklusen [CVZ10] observed that a randomized version of the pipage rounding algorithm gives a negatively correlated distribution, so it can be used instead of the rounding by sampling method to get an $O(\log(n)/\log\log(n))$ approximation for ATSP.

In the basic version we start from a fractional point $x$ in the matroid base polytope and we round to an integral base $T$. The method will have two properties,

i) For any element $e$, $\mathbb{P}[e \in T] = x(e)$.

ii) Elements are negatively correlated, i.e., for any set of elements $S \subseteq E$,

$$\mathbb{P}[S \subseteq T] \leq \prod_{e \in T} x(e).$$

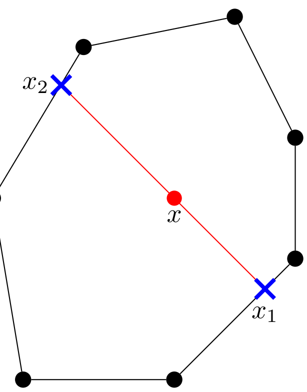


Figure 9.1: An iteration of the pipage rounding method. given the point $x$, we choose two variables say $x(e), x(f)$ and move randomly along the direction of the line that keeps the sum of $x(e), x(f)$ invariant until we hit the polytope, i.e., the blue crosses.

As we proved in lecture 3 the second property will imply that for any set $F \subseteq E$, the sum of indicator random variables of elements in $F$ is highly concentrated around its expected value. This, in addition to (i) are the only properties of the rounding by sampling method that we used to in the last lecture to show that a random spanning tree is $O(\log(n)/\log\log(n))$-thin. Therefore, one can use the pipage rounding method to get a different (and perhaps simpler) algorithm to round the solution of the Held-Karp relaxation to a thin spanning tree.

We say a set $S \subseteq E$ is tight with respect to a fractional point $x$ in the matroid base polytope if $x(S) = r(S)$. Before describing the algorithm we discuss a nice property of tight sets. We say $S, T \subseteq E$ cross if $S \setminus T, T \setminus S, S \cap T$ are nonempty. It is easy to see that if $S, T$ are crossing tight sets then $S \cap T, S \cup T$ are tight as well. This again follows by submodularity of the rank function. In particular, if $S, T$ are tight then,

$$\begin{aligned} x(S) + x(T) &= r(S) + r(T) \\ &\geq r(S \cap T) + r(S \cup T) \\ &\geq x(S \cap T) + x(S \cup T) \\ &= x(S) + x(T). \end{aligned}$$

The first inequality uses submodularity and the second inequality uses feasibility of $x$ in the matroid (base) polytope. So, all of inequalities must be equalities, in particular, $S \cap T$ and $S \cup T$ must be tight sets.

Now, we are ready to describe the pipage rounding algorithm. Given a nonintegral point $x$, we start by choosing a minimal tight set, say $T$ with at least two fractional elements say $0 < x(e), x(f) < 1$. We consider the line that is $+1$ in the direction of $x(e)$ and $-1$ in the direction of $x(f)$ and we randomly move forward or backward in this line, until we hit the faces of the polytope (see Figure 9.1). If we hit an integrality face, i.e., if either of $x(e)$ or $x(f)$ become integral then we have made progress. Otherwise, a new set, say $S$, becomes tight. As we keep $x(T)$ invariant, $T$ is still a tight set, so either $S \subset T$ or $S, T$ cross; in both of these cases $S \cap T$ is a new tight set that is a proper subset of $T$ with at least one fractional element. So, we run the same algorithm on $S \cap T$. We keep doing this procedure until we get an a new integral element in $x$. It is easy to see that after at most $|E| \cdot r(E)$ iterations we get to an actual integral point.

We didn't say how we move on the line $+x(e), -x(f)$. The idea is simple, say $x_1, x_2$ are the intersections of this line with the polytope (see Figure 9.1). With probability $p$ we go to $x_1$ and with probability $1 - p$ we go to $x_2$ for a value of $p$ where

$$px_1 + (1 - p)x_2 = x.$$

It is easy to see that after $i$-th step of the algorithm the the current (fractional) solution is the same as the starting point in expectation, so we preserve the marginal probabilities. In the next section we also show that elements are negatively correlated. This basically follows from the fact that whenever we increase a variable $x(e)$ we simultaneously decrease $x(f)$, so in this step, $e, f$ are negatively correlated and $e$ is independent of the rest of the elements.

## 9.3 Negative Correlation

In this section we show that pipage rounding method produces a negatively correlated distribution of the bases of a matroid. Let $X_i(e)$ be a random variable indicating the fractional point after the $i$-th step of the pipage rounding method. So, at the beginning $X_0(e) = x(e)$ with probability 1 and at the end of the algorithm it is an integral random variable. By the definition of the pipage rounding algorithm, $X_i(e)$ is indeed a martingale, its expectation is preserved throughout the algorithm, and we want to prove that they are negatively correlated. Fix a set $F$, all we need to show is that

$$\mathbb{E}\left[\prod_{e\in F} X_i(e)\right] \leq \prod_{e\in F} x(e). \tag{9.1}$$

We use an inductive proof. We show that for any $i$,

$$\mathbb{E}\left[\prod_{e\in F} X_{i+1}(e)\right] \leq \mathbb{E}\left[\prod_{e\in F} X_i(e)\right]. \tag{9.2}$$

In fact we prove a simpler claim, that is

$$\mathbb{E}\left[\prod_{e\in F} X_{i+1}(e)\middle| X_i(e)\right] \leq \prod_{e\in F} X_i(e). \tag{9.3}$$

Note that in the LHS we are conditioning on the whole vector $X_i$. Taking expectations from both sides of the above inequality implies (9.2). So, in the rest of this part we prove the above equation.

Say in step $i+1$ we move along the line we only change the variables corresponding to the elements $f_1, f_2$, i.e., we either increase $X_i(f_1)$ and decrease $X_i(f_2)$ or vice versa. Then, we for all $e \neq f_1, f_2$, $X_{i+1}(e) = X_i(e)$. In addition, we have the following two properties:

i) $\mathbb{E}\left[X_{i+1}(f_1)|X_i(f_1)\right] = X_i(f_1)$ and $\mathbb{E}\left[X_{i+1}(f_2)|X_i(f_2)\right] = X_i(f_2)$.

ii) $X_{i+1}(f_1) + X_{i+1}(f_2) = X_i(f_1) + X_i(f_2)$ with probability 1.

If $f_1, f_2 \notin F$, then (9.3) holds trivially. If $f_1 \in F$ and $f_2 \notin F$, then

$$\begin{aligned} \mathbb{E}\left[\prod_{e\in F} X_{i+1}(e)\middle| X_i\right] &= \mathbb{E}\left[X_{i+1}(f_1)|X_i\right] \cdot \prod_{e\in F\setminus\{f_1\}} X_i(e) \\ &= \prod_{e\in F} X_i(e), \end{aligned}$$

where the second equality uses property (i).

Perhaps the most interesting case is when $f_1, f_2 \in F$. In this case the negative correlation essentially follows from property (ii). First, note that similar to above all we need to show is that

$$\mathbb{E}\left[X_{i+1}(f_1) \cdot X_{i+1}(f_2) | X_i\right] \leq X_i(f_1) \cdot X_i(f_2). \tag{9.4}$$

Since the sum of $X_i(f_1) + X_i(f_2)$ is preserved with probability 1 we have

$$\mathbb{E}\left[(X_{i+1}(f_1) + X_{i+1}(f_2))^2 | X_i\right] = (X_i(f_1) + X_i(f_2))^2.$$

On the other hand, $X_i(f_1) - X_i(f_2)$ is preserved in expectation. Using the fact that for any random variable $Y$, $\mathbb{E}\left[Y^2\right] \geq \mathbb{E}\left[Y\right]^2$, we get

$$\mathbb{E}\left[(X_{i+1}(f_1) - X_{i+1}(f_2))^2 | X_i\right] \geq (X_i(f_1) + X_i(f_2))^2.$$

The difference of the above two equation proves (9.4). This was the last case, so we proved (9.3) and (9.1).

## 9.4 Conclusion

Chekuri, Vondrak and Zenklusen also introduce another algorithm that is similar to pipage rounding with some additional properties in some aspects, called the *randomized swap rounding*. They extend the above ideas to round a fractional point in the intersection of two matroids, e.g., a fractional matching in a bipartite graph. In this case it is impossible to round a fractional perfect matching into an integral one while satisfying the negative correlation property. To see this suppose $G$ is a cycle of length $2n$ and $x(e) = 1/2$ for all edges. Let $M_1$ be the odd edges and $M_2$ be the even edges of the cycle, i.e., $x = \mathbf{1}_{M_1}/2 + \mathbf{1}_{M_2}/2$ where $\mathbf{1}_F$ is the indictor vector of a set $F$ of edges. Observe that the only possible decomposition of $x$ as a convex combination of perfect matchings is $x = \mathbf{1}_{M_1}/2 + \mathbf{1}_{M_2}/2$. And, obviously in this decomposition edges within $M_1$ (or those within $M_2$) are positively correlated.

It is a very interesting to better understand the distribution of bases produced in a pipage rounding method. Although we have an algorithm to get a sample of this distribution we do not know how to analytically describe this distribution, e.g., is it possible to show that the distribution of spanning trees in the pipage rounding method is not a weighted uniform distribution of spanning trees. What are other properties of this distribution, does it satisfy negative association (see next two lectures for the definition of negative association)?

## References

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