Recent Advances in Approximation Algorithms **Spring 2015**

Problem Set 1

Deadline: April 8th in class

1) In this exercise we overview basic facts about eigenvalues of graphs.

   a) Show that if G is d-regular then the largest eigenvalue of the adjacency matrix of G is equal to d.

   b) Show that a d-regular graph is bipartite if and only if the smallest eigenvalue of the adjacency matrix is $-d$.

   c) Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix of a d-regular graph G. Show that $d - \lambda_1, \ldots, d - \lambda_n$ are the eigenvalues of the Laplacian of G.

2) In this exercise we want to prove the Cayley's theorem, that is the number of spanning trees of a complete graph on n vertices is n^{n-2}. Show that all of the non-zero eigenvalues of the Laplacian of a complete graph is n. Use this to show that a complete graph has n^{n-2} spanning trees.

3) Show that if $A \preceq B$ then the i-th eigenvalue of A is at most the i-th eigenvalue of B.

4) Observe that for any PSD matrix A, $\det(A) \geq 0$. This is because all eigenvalues of A are nonnegative and $\det(A)$ is just the product of the eigenvalues of A. A minor of a matrix $A \in \mathbb{R}^{n \times n}$ is a square submatrix $A_{S,T}$ where $|S| = |T|$ but S is not necessarily equal to T. We say A is totally positive if $\det(A_{S,T}) \geq 0$ for any square minor $A_{S,T}$ of A.

   a) Recall that a principal minor of A is a square submatrix $A_{S,S}$ for some $S \subseteq [n]$, i.e., a minor $A_{S,T}$ where $S = T$. Show that if A is PSD then $\det(A_S) \geq 0$ for all principal minors of A.

   b) Show that a PSD matrix is not necessarily totally positive.

   c) Use the Cauchy-Binet formula to show that for any pair of totally positive matrices $A, B \in \mathbb{R}^{n \times n}$, AB is also totally positive.

5) Say $G = (V, E)$ and $H = (V, E')$ are unweighted graphs on the same vertex set V.

   a) show that if $L_G \preceq L_H$, then for any $S \subseteq V$,

   $$|E(S, \overline{S})| \leq |E'(S, \overline{S})|,$$

   where $E(S, \overline{S}), E'(S, \overline{S})$ are the sets of edges in the cut $(S, \overline{S})$ in G, H respectively.

   b) Show that the converse of the above statement is not necessarily true, i.e., construct G, H such that $|E(S, \overline{S})| \leq |E'(S, \overline{S})|$ for all $S \subseteq V$ but $L_G \not\preceq L_H$.
   Bonus point: Construct an example where $|E(S, \overline{S})| \leq |E'(S, \overline{S})|$ for all $S \subseteq V$ but $L_G \not\preceq \Omega(n) L_H$.