Recent Advances in Approximation Algorithms **Spring 2015**

Assignment 2

Deadline: April 20th

You are free to collaborate to solve these assignments.

Problem 1) Show that for any matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$

Problem 2) Show that if A, B are PSD then $\text{trace}(AB) \geq 0$.

Problem 3) Show that the maximum degree of vertices of a uniform spanning tree of a complete graph is $\Theta(\log(n)/\log\log(n))$.

Hint: Use the Prüfer code, http://en.wikipedia.org/wiki/Prüfer_sequence

Problem 4) We say that an unweighted graph $G = (V, E)$ is k-edge connected if for any cut $(S, \overline{S})$, $|E(S, \overline{S})| \geq k$. Recall that by Menger's theorem a graph is k-edge-connected if for any pair of vertices s, t there are at least k edge disjoint paths from s to t.

a) Suppose G is connected and let $\epsilon = \max_{e \in E} \text{Reff}(e)$. Show that G is $1/\epsilon$-edge connected.

b) Show that the converse of the above is not true. Bonus point: construct a k-edge-connected graph with n vertices where $\max_{e \in E} \text{Reff}(e) \geq \Omega(n)/k$.

c) Show that for any simple unweighted k-edge-connected graph $G = (V, E)$ (with no parallel edges),

$$\sum_{einE} \text{Reff}(e)^2 \leq O(n/k).$$

Hint: Show that for any simple unweighted graph G, and any edge $e = \{u, v\}$, $\text{Reff}(e) \geq \frac{1}{d(u)+1} + \frac{1}{d(v)+1}$ where $d(u), d(v)$ are the degrees of u, v are respectively.

Problem 5) In this exercise we want to show that the effective resistance is convex over the space of all positive definite matrices. For a PD matrix $A \in \mathbb{R}^{V \times V}$ let

$$\text{Reff}_A(s, t) = b_{s,t}^{\intercal} A^{-1} b_{s,t}.$$

Show that for any two matrix $A, B \succ 0$,

$$\text{Reff}_{A+B/2}(s, t) \leq \frac{1}{2}(\text{Reff}_A(s, t) + \text{Reff}_B(s, t)).$$

Hint: Use the Schur complement which says that for a positive definite matrix A,

$$\begin{bmatrix} A & B \\ B^{\intercal} & C \end{bmatrix} \succeq 0 \text{ if and only if } C - B^{\intercal} A^{-1} B \succeq 0.$$

Problem 6) Prove that the expected size of the diameter of a uniform spanning tree of a complete graph on n vertices is $\Omega(\sqrt{n})$.

Hint: Use the Aldous,Broder algorithm for sampling a uniform spanning tree.