Recent Advances in Approximation Algorithms **Spring 2015**

Assignment 3

Deadline: May 11th

You are free to collaborate to solve these assignments.

Problem 1) Given a feasible solution x of the Held-Karp relaxation of ATSP and a partitioning $V_1, \dots, V_k$ of V. Design a polynomial time algorithm that finds an Eulerian subgraph $F \subseteq A$ such that for any $1 \leq i \leq k$, $|\delta^+_F(V_i)| \geq 1$ and that $c(F) \leq c(x)$. In the special case where each V_i is singleton, this problem is known as the cycle cover problem.

Problem 2) Carathéodory's theorem says that any feasible point of a linear program in $\mathbb{R}^n$ can be written as a convex combination of at most $n+1$ of its vertices. Use this theorem to show that any point in the interior of a bounded linear program (with finite number of vertices) can be written as a convex combination where each vertex has a positive coefficient.

Problem 3) In this question we want to show that a $2k$-dimensional hypercube has a $O(1/k)$-thin spanning tree. We say a graph $G = (V, E)$ has an ϵ-thin spanning tree if there is a tree T where for any $S \subseteq V$,

$$|T(S, \overline{S})| \leq \epsilon \cdot |E(S, \overline{S})|.$$

Note that this is a slightly different definition compared to one we talked about in class, here G is an integral graph. We can obtain such a graph from a fractional spanning tree z simply by choosing large enough number C such that $C \cdot z$ is (almost) integral and then add $Cz(u,v)$ parallel edges between each pair of vertices u, v. You get extra credit if you find a different construction of thin trees of a hypercube with a different proof.

a) In the class we alluded to the fact that testing a thinness of a tree is not an easy problem, we are not aware of a polynomial size certificate for exactly testing the thinness. In this part we give a certificate for upper bounding thinness. Suppose G has a set of cycles $C_1, C_2, \dots, C_\ell$ such that

- Each cycle has exactly one edge of T, and each edge of T is an at least β cycles.
- Each edge not in T is in at most α cycles.

Show that T is α/β-thin

b) Next, we construct a connected thin connected graph. We decompose H_{2k} into 2^k subcubes where each of them is uniquely determined by the first k bits, i.e., for $0 \leq x < 2^k$ we use H_x to denote all vertices where their k leftmost digits is x. Let $h = k/2$ (assume k is even) and define the weight function $w : [2^k] \to [h]$

$$w(b_{k-1} \dots b_1 b_0) = \sum_{i=0}^{k-1} b_i \cdot i \mod h.$$

It has been shown that H_k can be composed into $k/2$ edge disjoint Hamiltonian paths. Assuming this result let $T_1, \dots, T_h$ be these Hamiltonian paths. For each $0 \leq x < 2^k$ add all edges of $T_{w(x)}$ in H_x to T_1^*. Use part (a) to show that T_1^* is $O(1/k)$ thin with respect to H_{2k}.

c) Let H'_x be the k-dimensional of vertices where their k rightmost bits is x. Similar to case b construct T_2^* by including edges of $T_{w(x)}$ in H'_x to $T*_2$. Show that $T_1^* + T_2^*$ is a connected $O(1/k)$ thin subgraph of H_{2k}.

Problem 4) In this part we want to write the dual of

$$\begin{aligned} \min \quad & \log\det X^{-1} \\ \text{subject to} \quad & a_i^\intercal X a_i \leq 1, \quad \forall 1 \leq i \leq m. \end{aligned}$$

for vectors $a_1, \ldots, a_m \in \mathbb{R}^n$ over the space of PD matrices $X \in \mathbb{R}^{n \times n}$. This program is known as the minimum volume covering ellipsoid problem.

a) First show that the strong duality holds.

b) Show that for an invertible (symmetric) linear map $X + tB$,

$$\frac{d}{dt} \det(X + tB) = \det(X + tB)\, \mathrm{trace}((X + tB)^{-1} B).$$

c) Use (b) to show that $\nabla \log\det(X) = X^{-1}$.

d) Let λ_i be the Lagrange dual variable to the constraint $a_i^\intercal X a_i \leq 1$. Show that the dual is as follows:

$$\begin{aligned} \max \quad & \log\det\left(\sum_{i=1}^m \lambda_i a_i a_i^\intercal\right) + n - \sum_{i=1}^m \lambda_i \\ & \lambda \geq 0. \end{aligned}$$

You don't need to worry about the constraint $X \succ 0$ when you write the dual. This is possible because when we calculate the gradient we assume that the matrix X is in the domain of PD matrices.

Problem 5) Prove the Gauss-Lucas theorem, that is for any univariate polynomial $p(z)$ the roots of $\mathrm{d}p/\mathrm{d}z$ can be written as a convex combination of the roots of $p(z)$.

Problem 6) Fix a set of real numbers $\lambda_1, \ldots, \lambda_n$ and let t be an indeterminant. Use the properties of real stable polynomials to show that

$$\sum_{S \in \binom{n}{k}} \prod_{i \in S} (1 + t\lambda_i)$$

is real rooted.