Counting and Sampling **Fall 2017**

Problem Set 2

Deadline: Dec 9th in Canvas

1) Recall that for a graph G the independence polynomial is defined as follows:

$$\mathrm{ind}_G(z) = \sum_I z^{|I|}.$$

where the sum is over all independent sets of G. Let Δ be the maximum degree of G. In this problem we show that if $|z| \le \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$ then $\mathrm{ind}_G(z) \neq 0$. Using Barvinok's polynomial approximation technique this gives an algorithm to estimate the independence polynomial at any z where $|z| \le \frac{1}{\beta} \cdot \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$ for any constant $\beta > 1$.

a) Show that for a vertex v of G.

$$\frac{\mathrm{ind}_G(z)}{\mathrm{ind}_{G-v}(z)} = 1 + z\frac{\mathrm{ind}_{G-v-N_v}(z)}{\mathrm{ind}_{G-v}(z)},$$

where N_v is the set of neighbors of v. In this assignment we assume $\mathrm{ind}_{G-v}(z) \neq 0$; this can be justified by induction but we leave it out for simplicity.

b) **Optional:** Show that if $|1-z| \le 1/d$, then $|1-1/z| \le \frac{1}{d-1}$.

c) Prove by induction that if vertex v has degree at most $\Delta - 1$, then $\mathrm{ind}_G(z) \neq 0$ and

$$\left|1 - \frac{\mathrm{ind}_{G-v}(z)}{\mathrm{ind}_G(z)}\right| < \frac{1}{\Delta - 1}$$

when $|z| \le \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$.

Hint: First show that $|1 - \frac{\mathrm{ind}_G(z)}{\mathrm{ind}_{G-v}(z)}| < \frac{1}{\Delta}$, then use part (b) to conclude the induction. To show the former, use the following telescopic product:

$$\frac{\mathrm{ind}_{G-v-N_v}(z)}{\mathrm{ind}_{G-v}(z)} = \frac{\mathrm{ind}_{G-v-v_1}(z)}{\mathrm{ind}_{G-v}(z)} \cdots \frac{\mathrm{ind}_{G-v-v_1-\dots-v_k}(z)}{\mathrm{ind}_{G-v-v_1-\dots-v_{k-1}}(z)},$$

where $N_v = \{v_1, \dots, v_k\}$ is the set of neighbors of v.

d) Use the same idea to show that if all vertices of G have degree equal to Δ still $\mathrm{ind}_G(z) \neq 0$.

2) Let $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n$ be a real rooted polynomial. Design a polynomial time algorithm that for $\epsilon > 0$ given the coefficients $a_1, \dots, a_k$ for $k = O(\frac{\log n}{\epsilon})$ estimates the largest root of p in absolute value within $1+\epsilon$ multiplicative error, i.e., if $r_1, \dots, r_n$ are the roots of p, the algorithm returns a number q such that

$$(1-\epsilon)q \le \max_i |r_i| \le (1+\epsilon)q.$$

Hint: Recall that for all i,

$$a_i = \sum_{1 \le j_1 < j_2 < \dots < j_i \le n} r_{j_1} \dots r_{j_i}.$$

Also, recall the Newton identities.