Counting and Sampling	**Fall 2017**

Lecture 18: Gurvits's Technique: Deterministic ALG for Permanent

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

Recall that for a matrix $A \in \mathbb{R}^{n\times n}$ the permanent of A is defined as follows:

$$\text{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} A_{i,\sigma_i}.$$

In this lecture we will prove the following theorem.

Theorem 18.1 ([Gur06])**.** *There is a polynomial time algorithm that for any nonnegative matrix $A \in \mathbb{R}^{n\times n}$ gives an e^n approximation of* per(A).

The basic idea is to study the following polynomial:

$$p(z_1, \dots, z_n) = \prod_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} z_j. \tag{18.1}$$

Observe that the coefficient of $z_1, \dots, z_n$ in the above polynomial is exactly per(A).

So, the main question that we would like to study in this lecture is the following:

Problem 18.2. *Suppose $p(z_1, \dots, z_n)$ is a multivariate polynomial with nonnegative coefficients and suppose we can evaluate p exactly (or approximately) at any point $z_1, \dots, z_n$. For what families of polynomials can we approximate the coefficient of a monomial $\prod_i z_i^{\kappa_i}$?*

Although here we will mainly focus on the case that $\kappa_i = 1$ for all i as we will discuss in future the techniques naturally generalize to all monomials.

18.1 Estimating Coefficients of Multivariate Polynomials

Before studying the above question let us start with a univariate version of this question. Let $p(z)$ be a univariate polynomial and suppose we want to estimate the coefficient z, i.e.,

$$\frac{d}{dz} p(z)|_{z=0}.$$

We can simply use the definition of the derivative and write

$$p'(0) = \lim_{z\to 0} \frac{p(z) - p(0)}{z}.$$

But if we only have access to multiplicative approximation of p the above approach fails. Instead, we use the following approximation: $\inf_{z>0} \frac{p(z)}{z}$. We show that if p is a real rooted polynomial this gives a constant factor approximation to p.

Lemma 18.3. *For any univariate real-rooted polynomial $p(z)$ with non-negative coefficients we have*

$$\inf_{z>0} \frac{p(z)}{z} \geq p'(0) \geq \frac{1}{e} \inf_{z>0} \frac{p(z)}{z}.$$

Proof. The first inequality is obvious. So, we prove the second inequality. The main idea of the proof is that a univariate real rooted polynomial with nonnegative coefficients is log-concave. First we prove this fact and then we use it to prove the claim.

Let $r_1, \dots, r_n$ be the roots of p where n is the degree of p. So, we can write

$$p(z) = (z - r_1) \dots (z - r_n).$$

So,

$$\log p(z) = \sum_i \log(z = r_i).$$

Note that since all coefficients of p are nonnegative all roots of p are nonpositive, i.e., $r_i \leq 0$ for all i. Therefore, $\log(z - r_i)$ is well-defined for all $z > 0$.

To show that p is log-concave we need to show that the above function is concave, or equivalently that its second derivative is non-positive for all $z \geq 0$. It follows that

$$\log'' p(z) = \sum_i \frac{-1}{(z - r_i)^2}.$$

The latter is non-positive since r_i is a real number for all i.

Now, we use the above fact to lower bound $p'(0)$. First of all note that for any concave function f and any pair of points x, y we have

$$f(y) \leq f(x) + (y - x) f'(x).$$

Therefore, for all $z > 0$,

$$\log p(z) \leq \log p(0) + z \frac{p'(0)}{p(0)}.$$

So, for all $z > 0$,

$$\log \frac{p(z)}{z} \leq \log p(0) + z \frac{p'(0)}{p(0)} - \log z.$$

Therefore,

$$\inf_{z>0} \log \frac{p(z)}{z} \leq \inf_{z>0} \log p(0) + z \frac{p'(0)}{p(0)} - \log z.$$

But the function in the RHS is convex in z. Therefore, we can differentiate to find its minimum. The minimum is attained at $z = \frac{p(0)}{p'(0)}$. At such a z we have

$$\inf_{z>0} \log p(0) + z \frac{p'(0)}{p(0)} - \log z = \log p(0) + 1 - \log \frac{p(0)}{p'(0)} = \log p'(0) + 1.$$

Therefore,

$$-1 + \inf_{z>0} \log \frac{p(z)}{z} \leq \log p'(0).$$

Raising both sides to power e proves the claim. □

Note that the above lemma fails miserably if p is not real rooted; for example, if $p = z^2 + 1$, then

$$\inf_{z>0} \frac{z^2+1}{z} = \inf_{z>0} z + 1/z \geq 2,$$

whereas $p'(0) = 0$.

The main question that we would like to study in this lecture is a multivariate generalization of the above lemma. So, we need to find the right generalization of real-rooted polynomials.

First of all, considering the above lemma a natural generalization to a multivariate polynomial p is as follows:

$$\inf_{z_1,\dots,z_n>0} \frac{p(z_1,\dots,z_n)}{z_1\dots z_n} \geq \frac{\partial^n}{\partial z_1 \dots \partial z_n} p(z_1,\dots,z_n)|_{z_1=\dots=z_n=0} \geq e^{-n} \inf_{z_1,\dots,z_n>0} \frac{p(z_1,\dots,z_n)}{z_1\dots z_n}. \quad (18.2)$$

The LHS obviously holds for any polynomial p with non-negative coefficients.

We would like to prove the the LHS inductively for a class of Ω-stable polynomials. Later on we will see that it is enough to let $\Omega = \mathbb{H}$ be upper half complex plane:

$$\mathbb{H} = \{c \in \mathbb{C} : \Im c > 0\}. \quad (18.3)$$

Suppose that for any Ω-stable polynomial, of $n-1$ variables (18.2) holds, and say we want to prove this inequality for a polynomial p with n variables. Naturally, to use induction we need closure of Ω-stable polynomials under differentiation. So, let

$$q(z_1,\dots,z_{n-1}) = \frac{\partial p(z_1,\dots,z_n)}{\partial z_n}\Big|_{z_n=0}.$$

Then, by induction hypothesis,

$$\frac{\partial^{n-1}}{\partial z_1 \dots \partial z_{n-1}} q(z_1,\dots,z_{n-1})\Big| z_1 = \dots = z_{n-1} = 0 \geq e^{-(n-1)} \inf_{z_1,\dots,z_{n-1}>0} \frac{q(z_1,\dots,z_{n-1})}{z_1\dots z_{n-1}} \quad (18.4)$$

Now observe that

$$\frac{\partial^n p(z_1,\dots,z_n)}{\partial z_1 \dots \partial z_n}\Big|_{z_1=\dots=z_n=0} = \frac{\partial^{n-1}}{\partial z_1 \dots \partial z_{n-1}} q(z_1,\dots,z_{n-1})\Big|_{z_1=\dots=z_{n-1}=0}.$$

So, by (18.4) to prove (18.2) it is enough to show that

$$\inf_{z_1,\dots,z_{n-1}>0} \frac{q(z_1,\dots,z_{n-1})}{z_1\dots,z_{n-1}} \geq \frac{1}{e} \cdot \inf_{z_1,\dots,z_n>0} \frac{p(z_1,\dots,z_n)}{z_1\dots,z_n}.$$

To prove the above it is enough to show that for any fixed $z_1,\dots,z_{n-1} > 0$, there exists $z_n > 0$ such that

$$q(z_1,\dots,z_{n-1}) \geq \frac{1}{e} \cdot \inf_{z_n>0} \frac{p(z_1,\dots,z_n)}{z_n}. \quad (18.5)$$

Let $f(z) = p(z_1,\dots,z_{n-1},z)$. Then, observe that $z_1,\dots,z_{n-1}) = f'(0)$. So, we can rewrite the above as follows:

$$f'(0) \geq \frac{1}{e} \cdot \inf_{z.>0} \frac{f(z)}{z}.$$

This is exactly what we proved in Lemma 18.3. So, to prove the above all we need is that the polynomial $f(z)$ is a real rooted polynomial with nonnegative coefficients. Firstly, observe that $f(z)$ has nonnegative coefficients because all operations that we are doing on the original p is just differentiation and substitution with positive reals.

Furthermore, as we discussed before, for any set Ω, Ω-stable polynomials are closed under substitution of positive reals if positive reals are in the (closure of) Ω. So, here are the main properties that we need from Ω:

Differentiation: $\partial p / \partial z_1$,

Substitution: $p(c, z_2, \ldots, z_n)$ for $c > 0$,

Real-rootedness: Any univariate Ω-stable polynomial must be real rooted.

All these properties are satisfied by letting $\mathbb{H}$ be the upper-half complex plane as defined in (18.3). In particular, substitution hold because real numbers are in the bound of $\mathbb{H}$. Any univariate $\mathbb{H}$-stable polynomial with real coefficients is real rooted because the roots of any univariate polynomial come in conjugate pairs, i.e., z is a root if an only if $\bar{z}$ is a root.

It is not hard to see that these polynomials are also closed under differentiation. Therefore, these polynomials satisfy (18.5) and hence (18.2). The following theorem follows:

Theorem 18.4 ([Gur06]). *For any $\mathbb{H}$-stable polynomial $p(z_1, \ldots, z_n)$ with nonnegative coefficients,*

$$\inf_{z_1,\ldots,z_n>0} \frac{p(z_1,\ldots,z_n)}{z_1 \ldots z_n} \geq \frac{\partial^n}{\partial z_1 \ldots \partial z_n} p(z_1,\ldots,z_n)|_{z_1=\cdots=z_n=0} \geq e^{-n} \inf_{z_1,\ldots,z_n>0} \frac{p(z_1,\ldots,z_n)}{z_1 \ldots z_n}$$

18.2 Applications to Permanent

Now, we are ready to prove Theorem 18.1. Let p be the polynomial defined in (18.1). We claim that this polynomial is $\mathbb{H}$ stale. Therefore, by above theorem

$$\inf_{z_1,\ldots,z_n>0} \frac{\prod_{i=1}^n \sum_{j=1}^n A_{i,j} z_j}{z_1 \ldots z_n}$$

gives an e^n approximation to $\text{per}(A)$. To minimize the quantity in the infimum we need to convexify it. The idea is to do a change of variables, $z_i \to e^{y_i}$. Note that since $z_i > 0$, y_i will be un-constrainted. So, equivalently, we need to solve

$$\inf_{y_1,\ldots,y_n} \frac{\prod_{i=1}^n \sum_{j=1}^n A_{i,j} e^{y_j}}{e^{\sum_j y_j}}.$$

We show that the quantity inside the infimum is log-convex. This is because

$$\log \frac{\prod_{i=1}^n \sum_{j=1}^n A_{i,j} e^{y_j}}{e^{\sum_j y_j}} = \sum_{i=1}^n \left(\log \sum_{j=1}^n A_{i,j} e^{y_j} \right) - \sum_j y_j,$$

is convex. To see why the above quantity is convex recall that $\log(a_1 e^{x_1} + \ldots a_n e^{x_n})$ is convex if $a_1, \ldots, z_n \geq 0$.

To finish the proof of Theorem 18.1 all we need to show is that the polynomial p is $\mathbb{H}$-stable. Since $p(z_1, \ldots, z_n) = \prod_{i=1}^n \sum_j A_{i,j} z_j$, equivalently, it is enough to show that for all i, $\sum_j A_{i,j} z_j$ is $\mathbb{H}$-stable. We show that this is true if $A_{i,j} \geq 0$ for all j.

To prove it suppose $z_1, \ldots, z_n \in \mathbb{H}$, i.e., they all have positive imaginary value. Therefore, any positive combination of them also have positive imaginary value and it is nonzero. This completes the proof of Theorem 18.1.

We remark that the above approximation to permanent is improved to 2^n in a recent work of Gurvits and Samorodnitsky [GS14] using extensions of the above techniques. It is long standing open problem to design a deterministic $2^{o(n)}$ approximation algorithm to permanent of non-negative matrices.

References

[GS14] Leonid Gurvits and Alex Samorodnitsky. Bounds on the permanent and some applications. In *FOCS*, pages 90–99. IEEE Computer Society, 2014. 18-4

[Gur06] Leonid Gurvits. Hyperbolic polynomials approach to van der waerden/schrijver-valiant like conjectures: Sharper bounds, simpler proofs and algorithmic applications. In *STOC*, STOC '06, pages 417–426, 2006. 18-1, 18-4