CSE599s Counting and Sampling: Homework 1

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Problem 1 Solution

Let $D_{xy}(t) = \|K^t(x,\cdot) - K^t(y,\cdot)\|_{TV}$ and $D(t) = \max_{x,y\in\Omega} D_{xy}(t)$. For every t, since $\pi = \sum_{x\in\Omega}\pi(x)K^t(x,\cdot)$ is a convex combination of $K^t(x,\cdot)$ for $x \in \Omega$. We know that,

$$\Delta(t) = \max_{x\in\Omega}\|K^t(x,\cdot) - \pi\|_{TV} \leq \max_{x,y\in\Omega}\|K^t(x,\cdot) - K^t(y,\cdot)\|_{TV} = D(t).$$

On the other hand, we know that

$$D(t) = \max_{x,y\in\Omega}\|K^t(x,\cdot) - K^t(y,\cdot)\|_{TV} \leq 2\max_{x\in\Omega}\|K^t(x,\cdot) - \pi\|_{TV} = 2\Delta(t).$$

Let $X_0 = x$ and $Y_0 = y$. By the coupling lemma, there exists a coupling of $K^t(x,\cdot)$ and $K^t(y,\cdot)$ so that

$$\Pr[X_t \neq Y_t] = \|K^t(x,\cdot) - K^t(y,\cdot)\|_{TV} = D_{xy}(t).$$

Then for $s \geq 1$, we construct a coupling of X_{t+s} and Y_{t+s} as follows:

- If $X_t = Y_t$, then $X_{t+i} = Y_{t+i}$ for $i = 1, ..., s$.
- If $X_t = x'$ and $Y_t = y'$ where $x' \neq y'$, then by the coupling lemma we are able to construct X_{t+s} and Y_{t+s} so that

$$\Pr[X_{t+s} \neq Y_{t+s} \big| X_t = x', Y_t = y'] = \|K^s(x',\cdot) - K^s(y',\cdot)\|_{TV} = D_{x'y'}(s) \leq D(s).$$

By now we have that for every x, y

$$D_{xy}(t+s) \leq \Pr[X_{t+s} \neq Y_{t+s}] \leq D(s)D_{xy}(t) \leq D(s)D(t),$$

where the second inequality hold because $X_{t+s} \neq Y_{t+s}$ only when $X_t \neq Y_t$. Since the above inequality holds for every x, y, we have

$$D(t+s) = \max_{x,y\in\Omega} D_{x,y}(t+s) \leq D(s)D(t).$$

And thus $D(kt) \leq D(t)^k$ for any k.

Finally, by the relationship between $\Delta(t)$ and $D(t)$, we have

$$\Delta(\tau_{mix}\log(1/\epsilon)) \leq D(\tau_{mix}\log(1/\epsilon)) \leq D(\tau_{mix})^{\log(1/\epsilon)} \leq (2\Delta(\tau_{mix}))^{\log(1/\epsilon)} \leq (\frac{1}{e})^{\log(1/\epsilon)} = \epsilon.$$

Therefore, $\tau(\epsilon) \leq O(\tau_{mix}\log(1/\epsilon))$.

Reference

https://people.eecs.berkeley.edu/~sinclair/cs294/n3.pdf

Problem 2 Solution

We define a new sequence of random variables (Y_i) by $Y_i = X_i^2 - (1-\theta)i$. Then,

$$\begin{aligned}&\mathbb{E}[Y_{t+1}\big|X_0,...,X_t]\\=&\mathbb{E}[X_{t+1}^2-(1-\theta)(t+1)\big|X_0,...,X_t]\\=&\frac{1-\theta}{2}(X_t+1)^2+\frac{1-\theta}{2}(X_t-1)^2+\theta X_t^2-(1-\theta)(t+1)\\=&X_t^2-(1-\theta)t\\=&Y_t.\end{aligned}$$

Therefore, (Y_i) is a martingale with respect to (X_i).

For $i=-b,...,b$, let T_i to be the expected number of steps to first reach $+b$ or $-b$ starting from i. Then $T_b = T_{-b} = 0$ and $T_i = (\frac{1-\theta}{2}T_{i-1}+\frac{1-\theta}{2}T_{i+1}+\theta T_i)+1$ for $i=-b+1,...,b-1$. It is clear that this set of equations have a finite solution. Therefore $\mathbb{E}[T]=T_0$ is bounded. On the other hand, for all t before reaching $+b$ or $-b$ we have $\mathbb{E}[|Y_{t+1}-Y_t|\big|X_0,...,X_t] \le 2b+1+(1-\theta)$ which is bounded. Therefore, by the optional stopping theorem, we have

$$\mathbb{E}[X_T^2-(1-\theta)T]=\mathbb{E}[Y_T]=\mathbb{E}[Y_0]=0.$$

Thus we have

$$\mathbb{E}[T]=\frac{\mathbb{E}[X_T^2]}{1-\theta}=\frac{b^2}{1-\theta}.$$

Problem 3(a) Solution

Consider the following coupling between X_t and Y_t. Suppose that X_t and Y_t differ in d bits $i_1,...,i_d$, then with $\frac{1}{n+1}$ probability X_t remains unchanged, otherwise it uniformly flips a random bit. And,

- In all cases when X_t flips a bit on which X_t and Y_t agree, Y_t do the same.
- If d is an even number, then when X_t flips the bit i_k (on which they differ), Y_t flips the bit i_{d-k+1}. And when X_t remains unchanged, Y_t also remains unchanged.
- If d is an odd number, then for $k=1,...,d-1$ when X_t flips the bit i_k, Y_t flips the bit i_{d-k}. When X_t flips the last bit they differ, Y_t remains unchanged. And when X_t remains unchanged, Y_t flips the last bit they differ.

In the above coupling, the distance between X_t and Y_t decreases by 2 with probability $\frac{d}{n+1}$ when d is even and with probability $\frac{d+1}{n+1}$ when d is odd. Let T be the stopping time when X_T and Y_T agree. Then the expectation of T is bounded by

$$\max\{\sum_{k\ge1,2k-1\le n}\frac{n+1}{2k-1},\sum_{k\ge1,2k\le n}\frac{n+1}{2k}\}\le\frac{1}{2}n(\ln n+5).$$

Since the above time is highly concentrated around its expectation (for the same reason of the coupon collector problem) in the sense that

$$\Pr[T > \frac{1}{2}n(\ln n + 5) + cn] \leq e^{-c},$$

by Lemma 4.9 in Lecture Note 4, the mixing time τ_{mix} is upper bounded by $\frac{1}{2}n \ln n + O(n)$.

Reference

`https://people.eecs.berkeley.edu/~sinclair/cs294/n5.pdf`

Problem 4 Solution

Here we define the distance between two configurations σ^1 and σ^2 as $d(\sigma^1, \sigma^2) = \sum_i \mathbf{1}_{\sigma_i^1 \neq \sigma_i^2}$. Let X_t and Y_t be two configurations that differ only on site i on the torus (i.e. $d(X_t, Y_t) = 1$). And suppose that $X_t(i) = +$ and $Y_t(i) = -$. We consider the following coupling: X^t and Y^t pick the same uniformly random site and maximize the probability of agreement. Let j be the site X_t and Y_t pick. In this coupling, we have

- If $j = i$, then $d(X_{t+1}, Y_{t+1}) = 0$ with probability 1.
- If j is a neighbor of i. Let α be the number of $+$'s minus the number of $-$'s in the neighborhood of j in X_t (or Y_t) that excludes site i. Then we have

  $$\Pr[X_{t+1}(j) = +] = e^{2\beta(\alpha+1)}/(e^{2\beta(\alpha+1)} + e^{2\beta(-\alpha-1)}),$$

  and

  $$\Pr[Y_{t+1}(j) = +] = e^{2\beta(\alpha-1)}/(e^{2\beta(\alpha-1)} + e^{2\beta(-\alpha+1)}).$$

  Therefore

  $$\begin{aligned} &\Pr[d(X_{t+1}, Y_{t+1}) = 2] \\ =&\Pr[X_{t+1}(j) \neq Y_{t+1}(j)] \\ =&\Pr[X_{t+1}(j) = +] - \Pr[Y_{t+1}(j) = +] \\ =&\frac{e^{2\beta} - e^{-2\beta}}{e^{2\alpha\beta} + e^{-2\alpha\beta} + e^{2\beta} + e^{-2\beta}} \\ \leq&\frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta} + 2}, \end{aligned}$$

  where the last inequality is tight when $\alpha = 0$.

  Note that in this case when $d(X_{t+1}, Y_{t+1}) \neq 2$, we always have $d(X_{t+1}, Y_{t+1}) = 1$.
- If $j \neq i$ and j is not a neighbor of i then $d(X_{t+1}, Y_{t+1}) = 1$ with probability 1.

According to the above, the expected distance between X_{t+1} and Y_{t+1} is bounded by:

$$\mathbb{E}[d(X_{t+1}, Y_{t+1})] \leq 1 + \frac{4}{n} \cdot \frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta} + 2} - \frac{1}{n}.$$

When $\beta < 0.01$, $\frac{e^{2\beta}-e^{-2\beta}}{e^{2\beta}+e^{-2\beta}+2} < \frac{1}{8}$, and therefore $\mathbb{E}[d(X_{t+1}, Y_{t+1})] < 1 - \frac{1}{2n}$. Since the diameter of this metric is $O(n)$, by the path coupling lemma, the mixing time is bounded by $O(n \log n)$.

Reference

https://people.eecs.berkeley.edu/~sinclair/cs294/n20.pdf