Polynomial Paradigm in Algorithms **Winter 2020**

Lecture 10: Applications of Log Concave Polynomials

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In this lecture we will use the characterization theorems of completely log-concave polynomials from the last lectures to better study matroids and properties of log-concave polynomials. First, we note that similar to log-concave polynomials, completely log-concave polynomials are also closed with respect to affine transformations:

Theorem 10.1. *Let $p \in \mathbb{R}[z_1, \dots, z_n]$ be a completely log-concave polynomial and let $T : \mathbb{R}^m \to \mathbb{R}^n$ be an affine transformation defined as $x \mapsto Ax + b$ for $A \in \mathbb{R}^{n\times m}_{\geq 0}$, $b \in \mathbb{R}^n_{\geq 0}$. Then, $p(T(y_1, \dots, y_m))$ is completely log-concave.*

The proof of this simply follows from the same statement for log-concave polynomials and the fact that a directional derivative $D_v p(T)$ is the same as $(D_{Av}p) \circ T$.

10.1 Negative Correlation

We start with the Negative correlation property. As we explained in previous lectures, a uniform distribution over the bases of a given matroid is not negatively correlated. However, it can be seen that such a distribution is approximately negatively correlated.

Theorem 10.2. *Let $M = ([n], \mathcal{I})$ of rank r and let μ be the uniform distribution over the bases of M. Then, for any pair of elements i, j,*

$$2\mathbb{P}_\mu[i]\,\mathbb{P}_\mu[j] \geq \mathbb{P}_\mu[i,j].$$

Proof. As usual, let $g_\mu = \frac{1}{\#\text{Bases}}\sum_{B \text{ base}} z^B$ be the basis generating polynomial of μ. Let $Q = \nabla^2 g_\mu|_{z=\mathbf{1}}$. Since g_μ is log-concave, ,

$$0 \succeq \nabla^2 \log g_\mu|_{z=\mathbf{1}} = \frac{g_\mu(\mathbf{1}) \cdot Q - (\nabla g_\mu(\mathbf{1}))(\nabla g_\mu(\mathbf{1}))^T}{g_\mu^2(\mathbf{1})}.$$

Since $g_\mu(\mathbf{1}) = 1$, it follows that $Q \preceq \frac{r}{r-1}(\nabla g_\mu(\mathbf{1}))(\nabla g_\mu(\mathbf{1}))^T$. Note that for any i, j, $Q_{i,j} = \mathbb{P}[i,j]$, furthermore $Q_{i,i} = 0$ for any i.

Fix $1 \leq i, j \leq n$ and let $x \in \mathbb{R}^n$ be $x = \mathbf{1}_i/\mathbb{P}[i] + \mathbf{1}_j/\mathbb{P}[j]$. Then,

$$\begin{aligned}
2\frac{\mathbb{P}[i,j]}{\mathbb{P}[i]\,\mathbb{P}[j]} = \frac{Q_{i,j} + Q_{j,i}}{\mathbb{P}[i]\,\mathbb{P}[j]} = x_i(Q_{i,j} + Q_{j,i})x_j &= x^T Q x \\
&\leq (x\nabla g_\mu(\mathbf{1}))^2 = \left(\frac{\partial_{z_i} g_\mu(\mathbf{1})}{\mathbb{P}[i]} + \frac{\partial_{z_j} g_\mu(\mathbf{1})}{\mathbb{P}[j]}\right)^2 = \left(\frac{\mathbb{P}[i]}{\mathbb{P}[i]} + \frac{\mathbb{P}[j]}{\mathbb{P}[j]}\right)^2 = 4
\end{aligned}$$

as desired. □

Using the same idea we can show that $\mathbb{E}_{i,j}\left[\frac{\mathbb{P}[i,j]}{\mathbb{P}[i]\mathbb{P}[j]}\right] \leq \frac{n}{n-1}$. In other words, although matroids can have elements that are positively correlated a random pair of elements i, j are almost negatively correlated.

Here are two conjectures regarding negative correlation in Matroids that is interesting to study:

Conjecture 10.3. *Let $M = ([n], \mathcal{I})$ be a matroid, and let μ be the uniform distribution over the bases of M. Then,*

i) For any pair of elements i, j, $\frac{\mathbb{P}_\mu[i,j]}{\mathbb{P}_\mu[i]\mathbb{P}_\mu[j]} \leq 8/7$.

ii) For any element i,

$$\mathbb{E}_j\left[\frac{\mathbb{P}[i,j]}{\mathbb{P}[i]\,\mathbb{P}[j]}\right] \leq 1 + O(n).$$

10.2 Support of Log Concave Polynomials

Let $p = \sum_\kappa c(\kappa) z^\kappa \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$. Recall the support of p, denoted $\text{supp}(p)$, to be the set of κ for which $c_\kappa \neq 0$. Furthermore recall that $\text{Newt}(p)$ is the convex hull of all $\kappa \in \text{supp}(p)$. In this section we want to better understand the Newton polytope of multi-linear (completely) log-concave polynomials.

Proposition 10.4. *If $p = \sum_{S \subseteq [n]} c(S) z^S$ is log-concave then every edge of* $\text{Newt}(p)$ *is parallel to $\pm\mathbf{1}_i$ or $\mathbf{1}_i - \mathbf{1}_j$ for $i \neq j \in [n]$.*

If p is also homogeneous, then every edge of $\text{Newt}(p)$ is orthogonal to the all ones vector. So, in particular they cannot be $\pm\mathbf{1}_i$. We can combine this with the following useful characterization of the bases of a matroid. Given a collection of sets of size d, $\mathcal{S} \subseteq \binom{[n]}{d}$, let P_B denote the convex hull of the indicator vectors of all sets in B: $P_B = \text{conv}\{\mathbf{1}_S : S \in \mathcal{S}\}$.

Theorem 10.5 (Gelfand, Goresky, MacPherson and Serganova)**.** *$\mathcal{B}$ is the set of bases of a matroid if and only if all the edges of P_B are parallel to $\mathbf{1}_i - \mathbf{1}_j$ for some $i \neq j \in [n]$.*

The following statement follows:

Corollary 10.6. *The support of any homogenous multilinear log-concave polynomial correspond to bases of a matroid.*

However, unlike real stable polynomials, any matroid can be realized as a support of such a polynomial. So, we completely understand the support of these polynomials.

The above theorem amazingly defines a geometry for any matroid. Note that if instead of matroids we have had worked with linear matroids we would not have such a nice characterization of all polytopes with edges of the form $\mathbf{1}_i - \mathbf{1}_j$.

To prove this proposition, we need to understand another closure properties of log-concave polynomials.

Definition 10.7 (Initial Forms)**.** *For any $w \in \mathbb{R}^n$, we can define the degree p with respect to w,*

$$\deg_w(f) = \max_{\kappa \in \text{supp}(p)} \langle w, \kappa \rangle$$

Also, we define the initial form of p with respect to w,

$$\text{In}_w\, p = \sum_{\kappa : \langle w, \kappa \rangle = \deg_w(p)} c(\kappa) z^\kappa.$$

The following lemma is immediate:

Lemma 10.8. *If p is log-concave, then for any $w \in \mathbb{R}^n$, $\text{In}_w\, p$ is also log-concave.*

Proof. for $t \in \mathbb{R}_+$, define the polynomial

$$p_t(x) \;=\; t^{-\deg_w(p)} \cdot p(t^{w_1}x_1, \ldots, t^{w_n}x_n).$$

Note that the limit of p_t as $t \to \infty$ is exactly $\text{In}_w(p)$. Furthermore, since log-concave polynomials are closed under external fields $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_{\geq 0}$, and $p = p_1$ is log-concave on $(\mathbb{R}_+)^n$, p_t is log-concave for any t. The claim then follows from the fact that the class of log-concave polynomials is closed under taking limits. □

Proof of Proposition 10.4. Suppose $\mathbf{1}_S, \mathbf{1}_T \in \{0,1\}^n$ are vertices of an *edge* of $\text{Newt}(p)$ and let $w \in \mathbb{R}^n$ be a vector maximizing this edge, i.e., for any $\kappa \neq \mathbf{1}_S, \mathbf{1}_T$ where $\kappa \in \text{supp}(p)$ we have

$$\langle w, \kappa \rangle < \langle w, \mathbf{1}_S \rangle.$$

It then follows that $\text{In}_w\, p = c_S z^S + c_T z^T$. By Lemma 10.8, $\text{In}_w\, p$ is log-concave.

Note that $\mathbf{1}_S - \mathbf{1}_T = \mathbf{1}_{S \smallsetminus T} - \mathbf{1}_{T \smallsetminus S}$, so it suffices show that $S \smallsetminus T$ and $T \smallsetminus S$ both have size at most 1. We show the the latter and the former can be shown similarly.

Suppose for the sake of contradiction that there exists $i, j \in T \smallsetminus S$ such that $i \neq j$. Consider specializing $\text{In}_w\, f$ to $x_k = 1$ for all $k \in (S \cup T) \smallsetminus \{i, j\}$; call the result polynomial $q(z_i, z_j)$. It follows that $c_S + c_T z_i z_j$ is log-concave on $(\mathbb{R}_+)^2$. But then by Problem 4 of HW2 we must have $c_S \cdot c_T \leq 0$ which implies $c_S = c_T = 0$ (as they are non-negative), i.e., $\mathbf{1}_S, \mathbf{1}_T \notin \text{supp}(p)$ which is a contradiction. Therefore $|T \smallsetminus S| \leq 1$ as desired. □

10.3 Mason's Log Concavity Conjecture

Definition 10.9. *We say a sequence $a_1, \ldots, a_n$ of non-negative reals is log-concave if*

- *For any $1 < i < n$, $a_i^2 \geq a_{i-1} \cdot a_{i+1}$.*
- *It has no internal zeros; in other words, the support of a is an interval of $\mathbb{Z}$.*

We say this sequence is ultra *log-concave if the sequence $\frac{a_1}{\binom{n}{1}}, \ldots, \frac{a_n}{\binom{n}{n}}$ is log-concave. Note that ultra log-concavity implies log-concavity but not vice versa.*

In this section we prove the following conjecture of Mason:

Conjecture 10.10. *For any matroid $M = ([n], \mathcal{I})$ let a_i be the number of independent sets of rank i. Then, the sequence $a_1, \ldots, a_n$ is ultra log-concave.*

Log-concave sequences are always of interest in Mathematics. As alluded to in lecture 1, for any ultra log-concave sequence $a_1, \ldots, a_n$ there exists two convex bodies A, B in $\mathbb{R}^n$ such that $\sum_{i=0}^n a_i t^i = \text{vol}(A + tB)$. So, this defines a geometry for independence numbers of a matroid.

Before proving the conjecture, first we relate completely log-concave polynomials to ultra log-concave sequences.

Lemma 10.11. *Let $p \in \mathbb{R}_{\geq 0}[y,z]$ defined as*

$$p(y,z) = a_0 z^n + a_1 y z^{n-1} + \cdots + a_n y^n$$

be a n-homogenous completely log-concave polynomial. Then, for any $0 < i < n$,

$$\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{d}{k-1}} \cdot \frac{a_{k+1}}{\binom{n}{k+1}}.$$

Proof. Since p is completely log-concave, for any $1 < k < n$, the quadratic $q(y,z) = \partial_y^{n-k-1}\partial_z^{k-1} p$ is log-concave over $\mathbb{R}^2_{\geq 0}$. Notice that for any $0 \leq m \leq n$,

$$\partial_y^{n-m}\partial_z^m p \;=\; (n-m)!\; m!\; a_m \;=\; n!\; \frac{a_m}{\binom{n}{m}}.$$

Using this for $m = k-1, k, k+1$, we can write the Hessian of q as

$$\nabla^2 q \;=\; \begin{bmatrix} \partial_y^2 q & \partial_y \partial_z q \\ \partial_y \partial_z q & \partial_z^2 q \end{bmatrix} \;=\; n! \begin{bmatrix} a_{k-1} \Big/ \binom{n}{k-1} & a_k \big/ \binom{n}{k} \\ a_k \big/ \binom{n}{k} & a_{k+1} \Big/ \binom{n}{k+1} \end{bmatrix}.$$

Since q is log-concave on $\mathbb{R}^2_{\geq 0}$, $\nabla^2 q$ has exactly one positive eigenvalue. Therefore, $\det(\nabla^2 q) \leq 0$. This gives the desired inequality:

$$0 \;\geq\; \det(\nabla^2 q) \;=\; (n!)^2 \frac{a_{k-1}}{\binom{n}{k-1}} \cdot \frac{a_{k+1}}{\binom{n}{k+1}} - \left(\frac{a_k}{\binom{n}{k}}\right)^2$$

as desired. □

Let me first say a direction consequence of the above lemma:

Lemma 10.12. *Let $M = ([n], \mathcal{I})$ be a matroid and let μ be the uniform distribution over the bases of M. For any set S, the sequence $a_0, \ldots, a_n$ where $a_i = \mathbb{P}_{B \sim \mu}[|B \cap S| = i]$ is ultra log-concave.*

Proof. Let $g_\mu = \frac{1}{\#\text{Bases}} \sum_{B \text{ Base}} z^B$ be the generating polynomial of μ. Symmetrize g_μ as follows: substitute $z_i \leftarrow y$ for any $i \in S$ and $z_i \leftarrow z$ for any $i \notin S$, and let $q(y,z)$ be the resulting polynomials. It follows that

$$q(y,z) = \sum_{i=0}^{|S|} a_i y^i z^{|S|-i}$$

is a completely log-concave polynomial. So, the statement follows from the previous lemma. □

The above lemma shows that although the rank sequence of S is not necessarily a sum of independent Bernoulli random variables its distribution is highly concentrated similar to sum of Bernoullies. It turns out that the concentration is not limited to linear functionals. One can prove concentration of Lipschitz functions using connections to High dimensional expanders that we will explain later.

Theorem 10.13 (Cryan, Guo, Mousa)**.** *Let $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be probability distribution such that g_μ is d-homogeneous and completely log-concave. Let $f : [n] \to \mathbb{R}$ be a 1-Lipschitz function, i.e., for any two sets S, T of size d, $|f(S) - f(T)| \leq |S \Delta T|/2$. Then,*

$$\mathbb{P}_{S\sim\mu}\left[|f(S) - \mathbb{E}[f]| \geq \epsilon\right] \leq 2\exp\left(-\frac{\epsilon^2}{2d}\right).$$

This can be seen as a generalization of the strong concentratio

Finally, to prove Conjecture 10.10 we use the following theorem:

Theorem 10.14. *For any matroid $M = ([n], \mathcal{I})$, the polynomial*

$$I_M(z_1, \ldots, z_n, y) = \sum_{I \in \mathcal{I}} z^{n-|I|} z^I$$

is completely log-concave.

Note that this is a stronger fact than the basis generating polynomial being completely log-concave as we can get basis generating polynomial by taking $\partial_y^{n-r} I_M|_{y=0}$. But the proof is essentially the same we just need to prove that all quadratic partial derivates are log-concave polynomials.

Having this in hand, to prove Conjecture 10.10 all we need to do is to symmetrize $z_1, \ldots, z_n$ with z. The resulting polynomial

$$\sum_{I \in \mathcal{I}} z^{|I} y^{n-|I|}$$

is bi-variate n-homogeneous and completely log-concave. So, the coefficients form an ultra log-concave sequence.

This also shows that we can not characterize completely log-concave polynomials by the structure of their roots.