Lecture 2: Real Stable Polynomials

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

A multivariate polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ is *$\mathcal{H}$-stable* (or stable for short) if $p(z_1, \dots, z_n) \neq 0$ whenever $(z_1, \dots, z_n) \in \mathcal{H}^n$ where

$$\mathcal{H} = \{c \in \mathbb{C} : \Im(c) > 0\}$$

is the upper-half of the n-dimensional complex plane. We say p is *real* stable if all coefficients of p are real. Unless otherwise specified, all polynomials that we work with in this course have real coefficients.

Fact 2.1. *A univariate polynomial $p \in \mathbb{R}[t]$ is real rooted iff it is real stable.*

This simply follows from the fact that the roots of p come in conjugate pairs. So, if p has a root t with $\Im(t) < 0$, we have $\bar{t}$ is also a root with $\Im(\bar{t}) > 0$.

The above definition can be hard to understand; so, instead we discuss an equivalent definition.

Lemma 2.2. *A multivariate polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ is real stable iff for every point $a \in \mathbb{R}^n_{>0}$ and $b \in \mathbb{R}^n$, the univariate polynomial $p(at + b)$ is not identically zero and is real rooted.*

For example, $z_1 - z_2$ is not real stable because for $a = (1, 1)$ and $b = (0, 0)$

Proof. $\Rightarrow$: Fix $a \in \mathbb{R}^n_{>0}$ and $b \in \mathbb{R}^n$. If $p(at + b)$ is identically zero, then for $z_j = a_j i + b_j$, $p(z_1, \dots, z_n) = 0$ so p is not real stable. Otherwise, say $p(at + b)$ has a root t with $\Im(t) \neq 0$. Then, since $p(at + b)$ has real coefficients by Lemma 1.2 (see first lecture), we can assume $\Im(t) > 0$. Write $t = ci + d$; then for

$$z_j = a_j t + b_j = a_j ci + b_j + da_j$$

$p(z_1, \dots, z_n) = 0$ so p is not real stable.

$\Leftarrow$: Suppose p is not real stable; then there exists $(z_1, \dots, z_n) \in \mathcal{H}^n$ that is a root of p. Set $a_j = \Im(z_j)$ and $b_j = \Re(z_j)$ then $a_j > 0$ for all j so $p(at + b)$ is not identically zero and it must be real rooted. But $t = i$ is a root of $p(at + b)$ which is a contradiction. □

See Figure 2.1 for applications of the above lemma.

Let us discuss several examples of real stable polynomials

Linear Functions: A linear polynomial $p = a_1 z_1 + \dots + a_n z_n$ is real stable iff $a_1, \dots, a_n \geq 0$. To see this note that if all z_i have positive imaginary value then any positive combination also has a positive imaginary value and thus is non-zero.

Elementary Symmetric Polynomial: For any n and any k the elementary symmetric polynomial $e_k(z_1, \dots, z_n) = \sum_{S \in \binom{n}{k}} \prod_{i \in S} z_i$ is real stable. I leave this as an exercise.

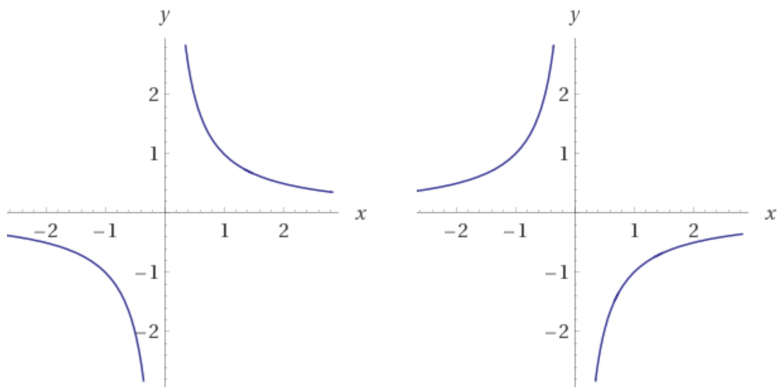

Figure 2.1: Left diagram shows zeros of the polynomial $1 - xy$ and the right diagram shows zeros of $1 + xy$ in the plane $\mathbb{R}^2$. Note that in the left figure any line pointing to the positive orthant crosses the zeros at two points so $1 - xy$ is real stable but this does not hold in the right figure so $1 + xy$ is no real stable.

Non-example The polynomial $z_1^2 + z_2^2$ is not real stable; for example let $z_1 = e^{\pi i/4}$ and $z_2 = e^{3\pi i/4}$.

One of the most important family of real-stable polynomials is the determinant polynomial.

Lemma 2.3. *Given PSD matrices $A_1, \ldots, A_n \in \mathbb{R}^{d\times d}$ and a symmetric matrix $B \in \mathbb{R}^{d\times d}$, the polynomial*

$$p(z) = \det\left(B + \sum_{i=1}^{n} z_i A_i\right)$$

is real stable.

Proof. By Lemma 2.2, it is enough to show that for any $a \in \mathbb{R}^n_{>0}$ and $b \in \mathbb{R}^n$

$$p(b + ta) = \det\left(B + \sum_{i=1}^{n} b_i A_i + t \sum_{i=1}^{n} a_i A_i\right)$$

is real-rooted. First, assume that $A_1, \ldots, A_n$ are positive definite. Then, $M = \sum_{i=1}^{n} a_i A_i$ is also positive definite. So, the above polynomial is real-rooted if and only if

$$\det(M)\det\left(M^{-1/2}\left(B + \sum_{i=1}^{n} b_i A_i\right)M^{-1/2} + tI\right)$$

is real-rooted. The roots of the above polynomial are the eigenvalues of the matrix $M' = M^{-1/2}(B + b_1 A_1 + \cdots + b_n A_n)M^{-1/2}$. Since $B, A_1, \ldots, A_n$ are symmetric, M' is symmetric. So, its eigenvalues are real and the above polynomial is real-rooted.

If $A_1, \ldots, A_n \succeq 0$, i.e., if the matrices have zero eigenvalues, then we appeal to the following theorem. This completes the proof of the lemma. In particular, we construct a sequence of polynomial with matrices $A_i + I/2^k$. These polynomials uniformly converge to p and each of them is real-stable by the above argument; so p is real-stable. $\square$

Lemma 2.4 (Hurwitz [**Hur95**])**.** *Let $\{p_k\}_{k\geq 0}$ be a sequence of Ω-stable polynomials over $z_1, \ldots, z_n$ for a connected and open set $\Omega \subseteq \mathbb{C}^n$ that uniformly converge to p over compact subsets of Ω. Then, p is Ω-stable.*

Definition 2.5 (d-homogeneous)**.** *A polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$ is d-homogeneous if $p(\lambda z_1, \ldots, \lambda z_n) = \lambda^d p(z_1, \ldots, z_n)$ for any $\lambda \in \mathbb{R}$.*

How general are these real stable polynomials and where should we look for them?

Theorem 2.6 (Choe, Oxley, Sokal, Wagner [**COSW02**])**.** *The support of any multi-affine homogeneous real stable polynomial corresponds to the set of bases of a matroid (more generally, the support corresponds to a jump system).*

For example, the support of a an elementary symmetric polynomial correspond to the set of bases of a uniform matroid whereas the non-stable polynomial $z_1 z_2 + z_3 z_4$ does not correspond to bases of a matroid.

2.1 Closure Properties

In general, it is not easy to directly prove that a given polynomial is real stable or a given univariate polynomial is real rooted. Instead, one may use an indirect proof: To show that $q(z)$ is (real) stable we can start from a polynomial $p(z)$ where we can prove stability using Lemma 2.3, then we apply a sequence of operators that preserve stability to $p(z)$ and we obtain $q(z)$ as the result.

In a brilliant sequence of papers Borcea and Brändén characterized the set of linear operators that preserve real stability [**BB09a**; **BB09b**; **BB10**]. We explain two instantiation of their general theorem and we use them to show that many operators that preserve real-rootedness for univariate polynomials preserve real-stability for of multivariate polynomials.

We start by showing that some natural operations preserve stability and then we highlight two theorems of Borcea and Brändén.

The following operations preserve stability.

Product If p, q are real stable so is $p \cdot q$.

Symmetrization If $p(z_1, z_2, \ldots, z_n)$ is real stable then so is $p(z_1, z_1, z_3, \ldots, z_n)$.

Specialization If $p(z_1, \ldots, z_n)$ is real stable then so is $p(a, z_2, \ldots, z_n)$ for any $a \in \mathbb{R}$. First, note that for any integer k, $p_k = p(a + \imath 2^{-k}, z_2, \ldots, z_n)$ is a stable polynomial (note that p_k may have complex coefficients). Therefore by Hurwitz theorem 2.4, the limit of $\{p_k\}_{k\geq 0}$ is a stable polynomial, so $p(a, z_2, \ldots, z_n)$ is stable.

External Field If $p(z_1, \ldots, z_n)$ is real stable then so is $q(z_1, \ldots, z_n) = p(\lambda_1 \cdot z_1, \ldots, \lambda_n \cdot z_n)$ for any positive vector $w \in \mathbb{R}^n_{\geq 0}$. If $q(z_1, \ldots, z_n)$ has a root $(z_1, \ldots, z_n) \in \mathcal{H}^n$ then $(\lambda_1 z_1, \ldots, \lambda_n z_n) \in \mathcal{H}^n$ is a root of p so p is not real stable.

Inversion If $p(z_1, \ldots, z_n)$ is real stable and degree of z_1 is d_1 then $p(-1/z_1, z_2, \ldots, z_n) z_1^{d_1}$ is real stable. This is because the map $z_1 \mapsto -1/z_1$ is a bijection between $\mathcal{H}$ and itself.

Differentiation If $p(z_1, \ldots, z_n)$ is real stable then so is $q = \partial p / \partial z_1$. This follows from Gauss-Lucas theorem. If $q(z_1, \ldots, z_n)$ is not real stable it has a root $(z_1^*, \ldots, z_n^*)$. Define $f(z_1) = p(z_1, z_2^*, \ldots, z_n^*)$. Then, $f'(z_1)$ has a root in $\mathcal{H}$. But the roots of $f'(z_1)$ are in the convex hull of the roots of $f(z_1)$ we get a contradiction because the complement of $\mathcal{H}$ is convex.

In the rest of this course we write ∂_{z_1} as a short hand for $\partial p/\partial z_1$.

We can continue this list and try to discover more and more closure properties. Borcea and Brändén proved a remarkable result characterizing all linear operators that are stability preserving. Here, we don't discuss their theorem in full generality but we discuss one of the main applications of their theorem which is being used in almost all applications.

Let $T : \mathbb{R}[z_1, \ldots, z_n] \to \mathbb{R}[z_1, \ldots, z_n]$ be an (differential) operator on polynomials with real coefficients defined as follows:

$$\sum_{\alpha,\beta \in \mathbb{N}_{\geq 0}} c_{\alpha,\beta} z^\alpha \partial^\beta$$

For example, $1 - z_1 \partial z_2$. In the above $z^\alpha = \prod_{i=1}^n z_i^{\alpha_i}$ and $\partial^\beta = \prod_{i=1}^n \partial^{\beta_i} z_i$.

Define $F_T \in \mathbb{R}[z_1, \ldots, z_n, w_1, \ldots, w_n]$

$$F_T(z_1, \ldots, z_n, w_1, \ldots, w_n) = \sum_{\alpha,\beta \in \mathbb{N}_{\geq 0}} c_{\alpha,\beta} z^\alpha (-w)^\beta.$$

Note that F_T is a polynomial with $2n$ variables.

Theorem 2.7. *T is an stability preserver operator, i.e., it maps any real stable polynomial to another real stable polynomial, iff F_T is real stable.*

For example, the operator $1 - \partial_{z_1}$ is stability preserver, because $1 + w_1$ is real stable. Also, $1 + z_2 \partial_{z_1}$ and $1 - \partial_{z_1} \partial_{z_2}$ are stability preserver.

For a non-example, $1+\partial_{z_1}\partial_{z_2}$ and $1-\partial_{z_1}\partial_{z_2}\partial_{z_3}$ are not stability preserver. This is because $(1+\partial_{z_1}\partial_{z_2})(z_1 z_2) = z_1 z_2 + 1$ is not real stable.

As a direct consequence we show that the Multi-Affine-Part (MAP) operator is stability preserving. Given a polynomial p, $\text{MAP}(p)$ zeros out every monomial of p with a square and keeps all multilinear monomials. For example,

$$\text{MAP}(1 + 2x + xy + x^2 y + z^3) = 1 + 2x + xy.$$

Lemma 2.8. *MAP is a stability preserving operator.*

Proof. Given a polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$. We can write MAP as the following differential operator

$$\text{MAP}(p) = \prod_{i=1}^n (1 - z_i' \partial_{z_i})|_{z_1 = \cdots = z_n = 0}.$$

Now, observe that $(1 - z_i' \partial_{z_i})$ is stability preserving. Furthermore, setting all $z_i = 0$ is also stability preserving. □

2.2 Applications

Given a graph $G = (V, E)$, our first application is to show that the matching polynomial

$$\mu_G(z_1, \ldots, z_n) = \sum_M (-1)^{|M|} \prod_{i \text{ sat in } M} z_i \tag{2.1}$$

is real stable. For observe that

$$p(z_1, \ldots, z_n) = \prod_{\{i,j\} \in E} (1 - z_i z_j)$$

is real stable. Now observe that $\mu_G = \mathrm{MAP}(p)$. So, μ_G is real stable.

As a consequence the univariate matching polynomial

$$\sum_M (-1)^{|M|} t^{|M|}$$

is real rooted. This follows by symmetrizing the polynomial in (2.1), i.e., set $z_i = t$ for all i and noting that any univariate real stable polynomial is real rooted.