Disclaimer: These notes have not been subjected to the usual scrutiny reserved for formal publications.

### 3.1 Gurvits’ Machinery

Given a matrix $M \in \mathbb{R}_{\geq 0}^{n \times n}$ the permanent of $M$ is defined as follows:

$$\text{per}(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} M_{i, \sigma_i}.$$  

This quantity is $\#P$-hard to compute exactly so one can only appeal to approximation algorithms. The problem has been studied for years. After so much effort, Jerrum, Sinclair and Vigoda [JSV04] managed to design a randomized $1 \pm \epsilon$ approximation algorithm for this problem that runs in time polynomial in $n, 1/\epsilon$.

It remains a fundamental open problem if one can design a deterministic algorithm with the same approximation factor. This question relates to several fundamental problem in TCS in the umbrella of polynomial identity testing (PIT). It is believed that one can simulation any polynomial time randomized algorithm with deterministic algorithm that also runs in polynomial time. Permanent is perhaps the most fundamental example that we have no clue how this can be done deterministically.

After a long line of works Linial, Samarodnitsky, Widgerson [LSW], Barvinok, Gurvits [Gur02], Gurvits, Samarodnitsky [GS14] and Anari, Rezaei [AR18] the best approximation factor that we know is $(\sqrt{2})^n$. I remark that if we can design a $2^n \epsilon$ approximation algorithm for any $\epsilon > 0$ then it is possible to improve the approximation factor $1 \pm \epsilon$.

Here, we prove the following result of Gurvits:

**Theorem 3.1** (Gurvits [Gur02]). Let $p \in \mathbb{R}[z_1, \ldots, z_n]$ be a real stable polynomial with non-negative coefficients. Then,

$$e^{-n} \inf_{z > 0} \frac{p(z_1, \ldots, z_n)}{z_1 \cdots z_n} \leq \partial_{z_1} \cdots \partial_{z_n} p|_{z=0} \leq \inf_{z > 0} \frac{p(z_1, \ldots, z_n)}{z_1 \cdots z_n}.$$  

In other words, the above theorem shows that we can approximate the coefficient of the monomial $z_1 \cdots z_n$ in the real stable polynomial $p$ (with non-negative coefficient) by the mathematical program $\inf_{z > 0} \frac{p(z_1, \ldots, z_n)}{z_1 \cdots z_n}$.

First I show that $\inf_{z > 0} \frac{p(z_1, \ldots, z_n)}{z_1 \cdots z_n}$ can be formulated as a convex programming problem. The idea is to do a change of variables $e^{y_i} \leftrightarrow z_i$. Note that since $z_i > 0$ this is a valid change of variables. Furthermore to minimize the ratio we can equivalently minimize the log of the ratio. So, we claim,

$$\log \frac{p(e^{y_1}, \ldots, e^{y_n})}{e^{y_1} \cdots e^{y_n}} = \log p(e^{y_1}, \ldots, e^{y_n}) - \sum_{i=1}^{n} y_i$$  

is convex. To see, it is enough to problem $\log p(e^{y_1}, \ldots, e^{y_n})$ is convex, as the second term is linear. To see
this, recall that log-sum-exp is a convex function i.e.,
\[ \log \sum_{i=1}^{n} a_i e^{(b_i y)} \]
is convex as long as \( a_1, \ldots, a_n \geq 0 \) and \( b_i \in \mathbb{R}^n \) for all \( i \).

Now, we show how to use this theorem to get an \( e^n \) approximation for permanent. Consider the following polynomial
\[ p(z_1, \ldots, z_n) = \prod_{i=1}^{n} \sum_{j=1}^{n} M_{i,j} z_j. \tag{3.1} \]

First, observe that \( p \) is real stable since all entries of \( M \) is nonnegative and product of real stable polynomials is real stable. Second, observe the coefficient of the monomial \( z_1 \ldots z_n \) is exactly \( \text{per}(M) \), i.e.,
\[ \partial_z \ldots \partial_z p |_{z=0} = \text{per}(M). \]

Therefore, Theorem 3.1 gives an \( e^n \) approximation algorithm to the \( \text{per}(M) \).

Finally, we use this theorem to prove the van-der-Waerden conjecture:

**Theorem 3.2.** Let \( M \in \mathbb{R}_{\geq 0}^{n \times n} \) be a doubly stochastic matrix, i.e., the sum of the entries in every row and column is exactly 1. Then, \( \text{per}(M) \geq e^{-n} \).

**Proof.** By Theorem 3.1 we can write
\[ \text{per}(M) \geq e^{-n} \inf_{z>0} \frac{p(z_1, \ldots, z_n)}{z_1 \ldots z_n} \]
where \( p \) is the same polynomial as in (3.1). So it is enough to show \( \inf_{z>0} \frac{p(z_1, \ldots, z_n)}{z_1 \ldots z_n} \geq \text{per}(M) \). To see this we use the weighted AM-GM inequality.

**Theorem 3.3** (Weighted AM-GM Inequality). Let \( a_1, \ldots, a_n \geq 0 \) and \( \lambda_1, \ldots, \lambda_n \geq 0 \) such that \( \sum \lambda_i = 1 \). Then,
\[ \sum \lambda_i a_i \geq \prod a_i^{\lambda_i}. \]

By the above theorem we can write
\[ p(z_1, \ldots, z_n) = \prod_{i=1}^{n} \sum_{j=1}^{n} M_{i,j} z_j \geq \prod_{i=1}^{n} \prod_{j=1}^{n} z_j^M_{i,j} = \prod_{j=1}^{n} \sum_{i=1}^{n} M_{i,j} = \prod_{j=1}^{n} z_j, \]
as desired. where the inequality follows by (weighted) AM-GM and the fact that every row of \( M \) adds up to 1 and the last equality follows by the fact that every column of \( M \) adds up to 1.

Before proving Theorem 3.1 let us first gain some intuition. The right side of the inequality is trivial and it holds for any polynomial with non-negative coefficients. This is because say \( p = \sum_{\alpha \in \mathbb{N}_{\geq 0}} c_{\alpha} z^\alpha \), then
\[ \inf_{z>0} \frac{p(z_1, \ldots, z_n)}{z_1 \ldots z_n} = c_{1,1,\ldots,1}^+ \inf_{z>0} \sum_{\alpha \in \mathbb{N}_{\geq 0}} c_{\alpha} \frac{z^\alpha}{z_1 \ldots z_n} \geq c_{1,1,\ldots,1}, \]
as desired.
Now, the left side of the inequality crucially uses the fact that \( p \) is real stable. For example, consider the non-stable polynomial \( z_1^2 + z_2^2 \). Then,
\[
\inf_{z > 0} \frac{z_1^2 + z_2^2}{z_1 z_2} = \inf_{z > 0} \frac{z_1}{z_2} + \frac{z_2}{z_1} \geq 2,
\]
where as the coefficient of \( z_1 z_2 \) is 0.

We start proof of Theorem 3.1 by proving a univariate version of the theorem. This not only will serve as the base of the induction, it will give some clues on how to approach the main theorem.

**Lemma 3.4.** For any real rooted polynomial \( f \in \mathbb{R}[t] \) with nonnegative coefficients,
\[
f'(0) \geq \frac{1}{e} \inf_{t > 0} f(t)/t.
\]

**Proof.** First, we show that \( f \) is a log-concave function.

**Fact 3.5.** Let \( f \in \mathbb{R}[t] \) be a real rooted polynomial with nonnegative coefficients then \( \log f \) is a concave function of \( R \geq 0 \).

This is because
\[
\log f = \log \prod_{i=1}^{n}(t + \alpha_i) = \sum_{i=1}^{n} \log(t + \alpha_i),
\]
where \(-\alpha_i\)'s are roots of \( f \). Since \( f \) has all positive coefficients, it is non-zero over positive reals so \( \alpha_i \geq 0 \) for all \( i \). The claim follows because \( \log \) is a concave function and sum of concave functions is concave.

Next, we prove the lemma. Since \( \log f \) is concave, for any \( t \geq 0 \) we can write,
\[
\log f(t) \leq \log f(0) + t \log f'(0).
\]

Therefore, for any \( t \geq 0 \) we get
\[
\frac{f(t)}{t} \leq \frac{f(0)}{t} + t \frac{f'(0)}{f(0)} - t
\]
Setting \( t = f(0)/f'(0) \) in the RHS we get
\[
\inf_{t > 0} \frac{f(t)}{t} \leq \log f(0) + 1 - \log \frac{f(0)}{f'(0)} = 1 + \log f'(0)
\]
as desired. \( \square \)

Having proved the lemma we are ready to prove Theorem 3.1. We prove by induction on \( n \). Let \( q(z_1, \ldots, z_{n-1}) = \partial_{z_n} p|_{z_n=0} \). Note that by closure properties of real stable polynomials \( q \) is a real stable polynomial. Furthermore, it has non-negative coefficients. Then,
\[
\partial_{z_1} \cdots \partial_{z_n} p|_{z=0} = \partial_{z_1} \cdots \partial_{z_{n-1}} q|_{z=0} \geq e^{-(n-1)} \inf_{z_1, \ldots, z_{n-1} > 0} \frac{q(z_1, \ldots, z_{n-1})}{z_1 \cdots z_{n-1}}
\]
where the inequality follows by IH. Say the infimum in the RHS is attained at a point \( z_1^*, \ldots, z_{n-1}^* \). Define \( f(z_n) = p(z_1^*, \ldots, z_{n-1}^*, z_n) \). Then, observe that
\[
q(z_1^*, \ldots, z_{n-1}^*) = f'(0) \geq \frac{1}{e} \inf_{z_n > 0} \frac{f(z_n)}{z_n} = \frac{1}{e} \inf_{z_n > 0} \frac{p(z_1^*, \ldots, z_{n-1}^*, z_n)}{z_n},
\]
\(^1\)to be more precise we need to do a \( \epsilon-\delta \) argument but here we avoid that for the simplicity of the argument.
where the inequality follows by the fact that $f$ is a real stable polynomial with non-negative coefficients. Therefore,

$$\partial z_1 \ldots \partial z_n p|_{z=0} \geq e^{-(n-1)} \frac{q(z_1^*, \ldots, z_{n-1}^*)}{z_1^* \ldots z_{n-1}^*} \geq e^{-n} \inf_{z_n > 0} \frac{p(z_1^*, \ldots, z_{n-1}^*, z_n)}{z_1^* \ldots z_{n-1}^* z_n} \geq e^{-n} \inf_{z > 0} \frac{p(z_1, \ldots, z_n)}{z_1 \ldots z_n}$$

where the first inequality follows by (3.2) and the second inequality follows by (3.3). This finishes the proof of Theorem 3.1.

**Remark 3.6.** The above proof is fairly general; it mainly works for polynomial whose univariate restrictions are log-concave. Gurvits has studied many generalizations and extensions of the above proof such as using it to estimate mixed volume of convex bodies. In the following lectures we will discuss a generalization of this proof and applications to problems in Economics and Game Theory.

### 3.2 Log Concavity of Real Stable Polynomials

An immediate generalization of the Lemma 3.4 is the following theorem:

**Theorem 3.7.** Let $p \in \mathbb{R}[z_1, \ldots, z_n]$ be a homogeneous real stable polynomial. Then, $p$ is log-concave over $\mathbb{R}_n^{\geq 0}$.

This theorem can be seen as a generalization of the well-known fact that $\det(P)$ is log-concave over the space of PSD matrices.

**Proof.** In the HW we will see that all coefficients of $p$ must be non-negative so $\log p$ is well defined over $\mathbb{R}_n^{\geq 0}$. It is enough to show that $\log p$ is concave along any interval in the positive orthant. Let $a, b \in \mathbb{R}_n^{>0}, b \in \mathbb{R}^n$, and consider the line $a + tb$ where for any $t \in [0, 1], a + tb \in \mathbb{R}_n^{>0}$. We show that $\log p(a + tb)$ is concave. Say $p$ is $k$-homogeneous, then

$$p(a + tb) = p(t(a/t + b)) = t^k p(a/t + b).$$

Since $a \in \mathbb{R}_n^{>0},$ and $p(\cdot)$ is stable, $p(at+b)$ is real rooted. Write $p(at+b) = p(a) \prod_{i=1}^k (t - \lambda_i)$ where $\lambda_1, \ldots, \lambda_k$ are the roots.

Then, we have

$$p(a/t + b) = p(a) \prod_{i=1}^k (1/t - \lambda_i).$$

So,

$$p(a + tb) = t^k p(a/t + b) = p(a) \prod_{i=1}^k (1 - t \lambda_i).$$

We claim that for all $1 \leq i \leq k$, $\lambda_i < 1$. Otherwise, for some $t \in [0, 1], p(a + tb) = 0$, but since $a + tb \in \mathbb{R}_n^{>0}, p(a + tb) > 0$ which is a contradiction. Therefore,

$$\log p(a + tb) = \log p(a) + \sum_{i=1}^k \log (1 - t \lambda_i).$$

The theorem follows by the fact that $\log(1 - t \lambda)$ is a concave function of $t$ for $t \in [0, 1]$ when $\lambda < 1$. $\square$

**Remark 3.8.** If $p$ is real stable but not homogeneous but it has positive coefficients, it is still log-concave over $\mathbb{R}_n^{\geq 0}$. It turns out that in this case one can homogenize $p$ and apply the above theorem.
3.3 Maximum Sub-determinant Problem

In the maximum sub-determinant problem we are given a PSD matrix $M \succeq 0$ and an integer $k$ we want to output a set $S \in \binom{[n]}{k}$ such that the determinant of the square-submatrix $M_{S,S}$ is maximized. Here, we give a simple proof of a recent result of Nikolov [Nik16] using the theory of real stable polynomials.

**Theorem 3.9.** There is a polynomial time randomized algorithm that gives a $e^k$ approximation to the sub-determinant maximization problem.

First we need to construct a real stable polynomial.

**Lemma 3.10.** For any $k \geq 0$, and any $M \succeq 0$, the following polynomial is real stable

$$\sum_{S \in \binom{[n]}{k}} \det(M_{S,S}) z^S.$$

**Proof.** Let

$$Z = \begin{bmatrix} z_1 & & \\ & z_2 & \\ & & \ddots \\ & & & z_n \end{bmatrix}$$

We use the following linear algebraic identity:

$$\det(Z + tM) = \sum_{k=0}^{n} \sum_{S \in \binom{[n]}{k}} t^k z^S \det(M_{S,S}).$$

The identity is a generalization of the fact that the $k$-th coefficient of characteristic polynomial of $M$ is the sum of the determinants of all square $k \times k$ minors of $M$.

By mother-of-all theorem, since $M \succeq 0$ and $Z$ is diagonal, $\det(Z + tM)$ is real stable. Therefore,

$$\partial^n x - k \det(Z + tM)|_{t=0} = k! \sum_{S \in \binom{[n]}{k}} z^S$$

is real stable. The lemma simply follows by the closer under of real stable polynomials under inversion. □

Since $\sum_{S \in \binom{[n]}{k}} z^S$ is homogeneous, it is log-concave over $\mathbb{R}_{\geq 0}^n$. To prove Theorem 3.9 we use the following convex program:

$$\begin{align*}
\max & \quad \log \sum_{S \in \binom{[n]}{k}} \det(M_{S,S}) x^S \\
\text{s.t.} & \quad \sum_i x_i = k \\
& \quad x_i \geq 0 \quad \forall 1 \leq i \leq k.
\end{align*}$$

(3.4)

Note that the above program is a relaxation of the problem as $x = \mathbb{I}[OPT]$ is a feasible solution. It follows that the objective function is at least OPT.

To round we construct the following distribution $\mu$ over $\{1, \ldots, k\}$ where $\mu(i) = x_i/k$. We generate $k$ samples from $\mu$, $i_1, \ldots, i_k$. If these are all distinct we output the set $\{i_1, \ldots, i_k\}$ otherwise we output nothing. We claim that this algorithm receives at least $e^{-k}$ fraction of OPT in expectation.
For every set \( S = \{i_1, \ldots, i_k\} \) the probability that we output \( S \) is

\[
\frac{x_{i_1}}{k} \cdot \frac{x_{i_2}}{k} \cdot \cdots \cdot \frac{x_{i_k}}{k} \cdot k! = \frac{x^S k!}{k^k} \approx x^{S} e^{-k},
\]

where the \( k! \) term comes because we can sample the \( k \) elements of \( S \) in any of the \( k! \) possible orders. Therefore,

\[
\mathbb{E}[\text{ALG}] = \sum_{S \in \binom{[n]}{k}} \mathbb{P}[\text{S sampled}] \det(M_{S,S}) = \sum_{S \in \binom{[n]}{k}} x^{S} e^{-k} \det(M_{S,S}) \geq \text{OPT} e^{-k}
\]

as desired.

### 3.4 Strongly Rayleigh Distributions

Let \( \mu : 2^{[n]} \rightarrow \mathbb{R}_{\geq 0} \) be a probability distribution. For \( X \sim \mu \), generating polynomial of \( \mu \) is defined as follows:

\[
g_{\mu}(z_1, \ldots, z_n) = \mathbb{E}[z^X] = \sum_{S \subseteq [n]} \mathbb{P}[X = S] z^S.
\]

For example, say \( B_1, B_2 \) are two independent Bernoullis with success probabilities \( p_1, p_2 \) respectively. Then, their generating polynomial is defined as follows:

\[
p_1 p_2 z_1 z_2 + p_1 (1 - p_2) z_1 + p_2 (1 - p_1) z_2 + (1 - p_1)(1 - p_2) = (p_1 z_1 + (1 - p_1))(p_2 z_2 + (1 - p_2))
\]

We say \( \mu \) is strongly Rayleigh (SR) if \( g_{\mu} \) is a real stable polynomial.

### 3.5 Negative Correlation

A probability distribution \( \mu : 2^{[n]} \rightarrow \mathbb{R}_{\geq 0} \) is negatively correlated if for any \( i, j \),

\[
\mathbb{P}[i|j] \leq \mathbb{P}[i].
\]

In this section we prove that any strongly Rayleigh distribution is negatively correlated.

**Theorem 3.11.** A multilinear polynomial \( p \in \mathbb{R}[z_1, \ldots, z_n] \) is real stable iff for any \( i, j \)

\[
\partial_{z_i} p(x) \cdot \partial_{z_j} p(x) \geq p(x) \cdot \partial_{z_i} \partial_{z_j} p(x),
\]

where \( x \in \mathbb{R}^n \).

Before proving this theorem, let us discuss its implications.

**Proof.** \( \Rightarrow \): Assume \( p \) is real stable. First, note that for any \( a > 0 \) and \( b \in \mathbb{R} \), by closure properties of real stable polynomials, \( p(at + b, z_2, \ldots, z_n) \) is real stable. Then the bivariate restriction

\[
g(s, t) = p(x_1, \ldots, x_{i-1}, s + x_i, \ldots, x_{j-1}, t + x_j, x_{j+1}, \ldots, x_n)
\]

is real stable. Since \( p \) is multilinear, \( g \) is multilinear and it can be written as

\[
g(s, t) = a + bs + ct + dst,
\]
where \( a = p(x) \), \( b = \partial_z p(x) \), \( c = \partial_z j p(x) \) and \( d = \partial_z \partial z_j p(x) \). Now, in HW1 we will see that this implies \( bc \geq ad \). Therefore, (3.5) 

\[ \iff \]

Conversely, assume that (3.5) holds for all \( i, j \). We prove by induction; suppose 

\[ p(z_1, \ldots, z_{n+1}) = q(z_1, \ldots, z_n) + z_{n+1} r(z_1, \ldots, z_n). \]

By IH, for any \( \alpha \in \mathbb{R} \), the polynomial \( p(z_1, \ldots, z_n, \alpha) \) is real stable or is identically zero. Similarly we have this for \( q, r \). First, suppose that for some \( \alpha \in \mathbb{R} \), \( q + x r \) is identically zero. Then, we can write 

\[ p = (z_{n+1} - \alpha)r(z_1, \ldots, z_n) \]

and thus real stable and we are done. Otherwise, we assume \( p(z_1, \ldots, z_n, \alpha) \) is real stable for any \( \alpha \in \mathbb{R} \). This implies that for any \( \alpha \in R \), and \( (z_1, \ldots, z_n) \in \mathcal{H}^n \), \( p(z_1, \ldots, z_n, \alpha) \neq 0 \). Writing, \( p = r(q/r + z_{n+1}) \) and using that \( q/r \) is continuous, it follows that either 

\[ \Im(q/r) > 0 \quad \forall (z_1, \ldots, z_n) \in \mathcal{H}^n, \text{ or} \]

\[ \Im(q/r) < 0 \quad \forall (z_1, \ldots, z_n) \in \mathcal{H}^n. \]

In the former case we get that \( p(z_1, \ldots, z_{n+1}) \) is real stable and we are done. In the latter case we get that \( \tilde{p}(z_1, \ldots, z_{n+1}) = p(z_1, \ldots, z_n, -z_{n+1}) \) is real stable. Using the forward direction of the theorem this implies the reverse of (3.5), for \( i, n + 1 \) we get 

\[ \partial_z p(x) \cdot \partial_{z_{n+1}} \tilde{p}(x) \geq \tilde{p}(x) \partial_z \partial_{z_{n+1}} \tilde{p}(x) \]

for any \( x \in \mathbb{R}^{n+1} \). This implies 

\[ \partial_z p(x_1, \ldots, x_n, -x_{n+1}) \cdot -\partial_{z_{n+1}} p(x_1, \ldots, -x_{n+1}) \geq p(x_1, \ldots, x_{n+1}) \cdot -\partial_z \partial_{z_{n+1}} p(x_1, \ldots, -x_{n+1}) \]

so 

\[ \partial_z p(x) \cdot \partial_{z_{n+1}} p(x) \leq p(x) \cdot \partial_z \partial_{z_{n+1}} p(x) \]

for any \( x \in \mathbb{R}^{n+1} \). Since this the inverse of (3.5) indeed we have equality above. This implies that for any \( 1 \leq i \leq n \), \( r \partial_z q = q \partial_z r \). But the latter implies that \( q \) is a multiple of \( r \) and therefore \( p \) is real stable.