Polynomial Paradigm in Algorithms **Winter 2020**

Lecture 7: Hyperbolic Polynomials

Lecturer: Shayan Oveis Gharan *Feb 3rd*

This lecture is based on a talk by Michel Goemans at Simons Program on "Geometry of Polynomials". Let $p \in \mathbb{R}[z_1, \ldots, z_n]$ be a homogeneous polynomial of degree d.

Definition 7.1 (Hyperbolic Polynomials)**.** *A polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$ is hyperbolic in direction $e \in \mathbb{R}^n$ with $p(e) \neq 0$ if for all $x \in \mathbb{R}^n$, the univariate polynomial $t \mapsto p(x - te)$ has only real roots. We typically call these real roots $\lambda_1(x), \ldots, \lambda_d(x)$.*

Example 7.1. *$p(z) = \prod_{i=1}^{d} z_i$ and direction $e = \mathbf{1}_n$. Also, works for all $e \in \mathbb{R}^n_+$. More generally, if we take product of linear forms $\prod_{i=1}^{d} \ell_i(x)$ and e is not a root of this then we get hyperbolic polynomial.*

Example 7.2. *Take the polynomial $p(x) = x_n^2 - \sum_{i=1}^{n-1} x_i^2$. and take $e = (0, 0, \ldots, 0, 1)$.*

In general we get $(x_n - t)^2 - \sum_{i=1}^{n-1} x_i^2 = t^2 - 2x_n t + x_n^2 - \sum_{i=1}^{n-1} x_i^2$ with discriminant

$$\Delta = (2x_n)^2 - 4x_n^2 + \sum_{i=1}^{n-1} x_i^2 \geq 0$$

as desired.

Example 7.3. *X is the space of symmetric $n \times n$ matrices and $e = I$ and $p(X) = \det(X)$ we get*

$$p(X - te) = \det(X - tI), .$$

Example 7.4. *Take $p(x) = \det\left(\sum_{i=1}^{n} x_i A_i\right)$ where A_i's are symmetric matrices and choose e such that $\sum_{i=1}^{n} e_i A_i = I$. Then,*

$$p(x - te) = \det\left(\sum_{i=1}^{n} (x_i - te_i) A_i\right) = \det\left(\sum_{i=1}^{n} x_i A_i - tI\right)$$

Note that for polynomial p to be hyperbolic in direction e, all we need is that $\sum_{i=1}^{n} e_i A_i \succ 0$, i.e., it does not have to sum up to identity.

Definition 7.2. *M is a regular matroid if it can be represented over the reals by a totally unimodular matrix. The generating polynomial of these matroids give another example.*

7.1 Hyperbolic Cones

Definition 7.3 (Hyperbolic Cone)**.** *Given a polynomial p hyperbolic in direction e, define*

$$K_+^{(p)} = \{x \in \mathbb{R} : \lambda_i(x) > 0, \forall i\}.$$

You can also define K_{++} to denote the closure of K_+.

In first example we have

$$K_+ = \{x : x_i > 0, \forall i\}$$

is the non-negative orthant cone. In this example, we also discussed the case of working with linear forms. In such a case we get the polyhedral cone.

In Example 7.2 this gives the so called second order cone, a.k.a., ice cream cone:

$$K_+ = \{x \in \mathbb{R}^n : x_n > 0, \sum_{i=1}^{n-1} x_i^2 < x_n^2\}$$

In Example 7.3 $K_+ = \{X : X \succ 0\}$.

In Example 7.4 $K_+ = \{x : \sum_{i=1}^n x_i A_i \succ 0\}$. In this case K_{++} is the spectrahedral cone.

Fact 7.4. *We always have $e \in K_+^{(p)}$*

This is because

$$p(e - te) = (1-t)^d p(e)$$

where we used the homogeneity of p. Since $p(e) \neq 0$, it has d positive roots.

Here is an exercise

Lemma 7.5 (Gärding)**.** *K_+ is the component of $\mathbb{R}^n \setminus \{x : p(x) = 0\}$ containing e.*

Here is an immediate connection to real stability. Recall that we said a polynomial p is real stable iff for any positive direction $e \in R^n_{>0}$ and $x \in \mathbb{R}^n$, $p(x - te)$ is not identically zero and it is real rooted. This latter implies that $p(e) \neq 0$. Therefore, if p is also homogeneous, then it is hyperbolic with $K_+^{(p)} \supseteq \mathbb{R}^n_{>0}$.

The following theorem is a fundamental theorem of the field of hyperbolic programming.

Theorem 7.6 (Gärding)**.** *K_+ is convex.*

This implies that hyperbolic cones can be seen as a single generalization of polyhedral cones, second order cones and spectrahedral cones.

Proposition 7.7. *If $f \in K_{++}(e)$ then $p(x)$ is also hyperbolic in direction f; furthermore, $K_{++}(e) = K_{++}(f)$.*

Assume this proposition for now and we prove Theorem 7.6 using it. Suppose $e, f \in K_+$ and consider the convex combination $x = \lambda e + (1-\lambda) f$ and we need to show that $x \in K_+$, i.e., we need to show roots of $p(x - te)$ are positive. Write,

$$p(x - te) = p(\lambda e + (1-\lambda) f - te) = (1-\lambda)^d p\left(f - \frac{t-\lambda}{1-\lambda} e\right)$$

Since $f \in K_+(e)$, the roots of $s \mapsto p(f - se)$ are positive. Let t be an arbitrary root of $p(x - te)$; then we know $\frac{t-\lambda}{1-\lambda}$ is a root of $p(f - se)$, so $\frac{t-\lambda}{1-\lambda} > 0$. But, since $0 < \lambda < 1$ the latter implies that

$$t - \lambda > 0 \Leftrightarrow t > \lambda > 0$$

as desired.

Another view Write

$$p(x + te) = p(e)\left(t^d + a_1(x) t^{d-1} + \cdots + a_d(x)\right)$$

Since the roots are $\lambda_1(x), \ldots, \lambda_d(x)$ we can equivalently write

$$p(x + te) = p(e) \prod_{i=1}^{d} (\lambda_i(x) + t) = p(e) \left(t^d + \sigma_1(\lambda(x))t^{d-1} + \ldots\right).$$

where σ_i is the $i - th$ elementary symmetric polynomial in λ_i's. This in particular implies $a_i(x) = \sigma_i(x)$ for all i.

Theorem 7.8 (Renegar)**.** *$K_+ = \{x \in \mathbb{R}^n : a_i(x) > 0, \forall i\}$.*

Proof. First we show $K_+ \subseteq .$; if we take a point $x \in K_+$ all the roots of $p(x - te)$ are positive, so all roots of $p(x + te)$ are negative and all coefficients of $p(x + te)$ are positives.

Conversely, given x such that $a_1(x), \ldots, a_d(x) > 0$ we get that all roots of the polynomial $\frac{p(x+te)}{p(e)} = t^d + a_1(x)t^{d-1} + \cdots + a_d(x)$ are negative. So, the roots of $p(x - te)$ are all positive. □

Furthermore, this implies K_+ is a finite intersection of polynomially many polynomial equations so it is a semi-algebraic set.

7.2 Hyperbolic Optimization

For a hyperbolic cone $K_{++} \subseteq \mathbb{R}^n$,

$$\begin{aligned} \min \quad & \langle c, x\rangle \\ \text{s.t.,} \quad & \langle a_i, x_i\rangle = b_i \quad \forall i \\ & x \in K_{++} \end{aligned} \tag{7.1}$$

Any class of convex programming that we are aware of fits into this framework including linear programming, second order cone programming, semidefinite programming.

One can study properties of hyperbolic polynomials: Any hyperbolic polynomial p is log-concave in its hyperbolic cone. In fact the $\log p$ can be used as a barrier function for interior point methods applied to hyperbolic cone. Furthermore, one can study closure properties of hyperbolic polynomials. For example, given a hyperbolic polynomial p with cone K_+, if we differentiate p along any direction in K_+, the new polynomial is hyperbolic with respect to a super-set of K_+.

A d-homogenous polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$ is called Hermitian determinantal if there exists $d \times d$ symmetric matrices $M_1, \ldots, M_n$ such that

$$p = \det(\sum_i M_i).$$

As we discussed above if $\sum_i e_i M_i \succ 0$ then p is hyperbolic in direction e. Helton and Vinnokov in a remarkable result prove a converse to this in three dimensions:

Theorem 7.9. *Let $p \in \mathbb{R}[x, y, z]$ be hyperbolic in direction $e \in \mathbb{R}^3$. Then, there exists real symmetric matrices A, B, C such that $p = \det(xA + yB + zC)$ and $e_1 A + e_2 B + e_3 C \succ 0$.*

In other words, any hyperbolic polynomial in 3 variables is determinantal. To put it differently, any 3-dimensional hyperbolic cone is spectrahedral.

It is well-known that the above theorem does not naturally generalize to more than 3 variables, i.e., there are hyperbolic polynomials that are not determinantal such as the elementary symmetric polynomial. Nonetheless, a fundamental open problem is if every hyperbolic cone is spectrahedral. This is known as the generalized

LAX conjecture. To put it differently, the Generalized Lax Conjecture asserts that hyperbolic programming and semidefinite programming have the same feasible sets.

Conjecture 7.10 (Generalized Lax Conjecture)**.** *Let $p \in \mathbb{R}[z_1, \dots, z_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. Then there is a hyperbolic polynomial $q \in \mathbb{R}[z_1, \dots, z_n]$, such that $K_+^{(p)}(e) \subseteq K_+^{(q)}(e)$ and such that $p \cdot q$ has a definite determinantal representation, i.e.*

$$q \cdot h = \det(z_1 A_1 + \cdots + z_n A_n),$$

where $A_1, \dots, A_n$ are symmetric matrices with real entries and $\sum_i e_i A_i \succ 0$.

7.3 Application: Homogenization

To this date there are not much known applications of hyperbolic programming in theory of computing. It is imaginable that hyperbolic programming can be used to design improved approximation algorithms for fundamental optimization problems in the same way that Goemans-Williamson used semidefinite programming for the Max-Cut problem three decades ago.

In this section, we discuss an application of hyperbolic polynomials in proving closure under homogenization for real stable polynomials.

Theorem 7.11. *Given a real stable polynomial $p \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]$, the homogeneous polynomial $p_H = z_{n+1}^{\deg p} p(z_1/z_{n+1}, \dots, z_n/z_{n+1})$ is real stable.*

We prove this theorem in two steps: Let $e \in \mathbb{R}^{n+1}$ such that $e_1, \dots, e_n > 0$. First, we argue that for any real stable polynomial p, p_H is hyperbolic in direction e and then we argue that if p also has non-negative coefficients, then p_H is indeed real stable.

Step 1: p_H is hyperbolic in direction e: For the sake of contradiction, suppose there exists $x \in \mathbb{R}^{n+1}$ such that $p_H(x - te)$ has a non-real root t. Since $p_H(x - te)$ has real coefficients, both $t, \bar{t}$ are roots; we assume $\Im(t) > 0$ and $\Im(\bar{t}) < 0$. First, assume $x_{n+1} \neq 0$. Then, if $x_{n+1} > 0$, then $z = (x - te)/x_{n+1}$ is a root of p and $z \in \mathcal{H}^n$ which is a contradiction. Otherwise, if $x_{n+1} < 0$, then $z = (x - \bar{t}e)/x_{n+1}$ is a root of p and $z \in \mathcal{H}^n$. Finally, the case that $x_{n+1} = 0$ can be taken care of by the previous case and a limiting argument.

Step 2: Now, suppose p_H is hyperbolic in direction e. Then, by Proposition 7.7, p_H is hyperbolic in any direction f in $K_{++}(e)$ But since p_H has non-negative coefficients, every e in the positive orthant is in the same component as the vector e. Therefore, $K_{++}(e)$ has the positive orthant. Therefore, p_H is hyperbolic with respect to any $e \in \mathbb{R}^{n+1}_{\geq 0}$. But, this implies that p_H is real stable as desired.