Polynomial Paradigm in Algorithms **Winter 2020**

Lecture 8: Log Concave Polynomials and Matroids

Lecturer: Shayan Oveis Gharan *Feb 7th*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In this lecture, we study a generalization of real stable polynomials called (completely) log-concave polynomials. As we discussed earlier, we say a polynomial p is log-concave over $\mathbb{R}^n_{>0}$ if $\log p$ is a concave function over the positive orthant. Say p is a homogeneous polynomial. Since we need $\log p$ to be well-defined over the positive orthant, we need all coefficients of p to be non-negative. In lecture 3 we proved that real-stable polynomials with non-negative coefficients are log-concave. It turns out that the converse is not true; there are many log-concave polynomials which are not real stable. In the second part of this course we study the class of log-concave polynomials without looking at the geometry of the roots.

8.1 Quadratic Log Concave Polynomials

Let $q \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]$ be a 2-homogeneous polynomial, i.e., $q = zQz$ for some $Q \in \mathbb{R}^{n\times n}_{\geq 0}$. It turns out that quadratic log-concave polynomials are real stable.

Lemma 8.1. *The following are equivalent:*

i) q is log-concave.

ii) Q is negative semidefinite on $(Qa)^\perp = \{c : c^T Qa = 0\}$ for any $a \in \mathbb{R}^n_{\geq 0}$ where $q(a) \neq 0$.

iii) Q is negative semidefinite on $(Qa)^\perp$ for some $a \in \mathbb{R}^n_{\geq 0}$ where $q(a) \neq 0$.

iv) Q is negative semidefinite on some linear space of dimension $n-1$, i.e., Q has at least $n-1$ negative eigenvalues.

v) q is real stable.

First we prove (i)⇒(ii). Suppose q is log-concave and let $a \in \mathbb{R}^n_{\geq 0}$ such that $q(a) \neq 0$. So, $q(a) > 0$. Then,

$$\nabla^2 \log q(a) = \frac{q(a) \cdot \nabla^2 q - \nabla q(a) \nabla q(a)^T}{q(a)^2} \preceq 0.$$

Observe that $\nabla q(a) = 2Qa$. Note that since $a^T Qa \neq 0$, Qa is not the zero vector. Therefore, For $c \in (Qa)^\perp$, we can write

$$c^T(\nabla^2 \log q(a))c = \frac{c^T Qc}{q(a)} \leq 0$$

where we used that $q(a) > 0$, as desired.

(ii)⇒(iii) is immediate. (iii)⇒(iv) is immediate. (v)⇒(i) follows by the lemma in lecture (3). It remains to prove (iv)⇒(v):

By variational characterization Q has at least $n-1$ non-positive eigenvalues. On the other hand, since all entries of Q are non-negative (and Q is not zero), Q has at least one positive eigenvalue. In the following lemma we show that for such a Q, the polynomial zQz is real stable.

Lemma 8.2. *Let $q \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]$ be a 2-homogeneous polynomial. If $Q = \nabla^2(q)$ has exactly one positive and $n-1$ negative eigenvalues, then q is real stable.*

Note that by a limiting argument q is also real stable if it has exactly one positive (and $n-1$ non-positive) eigenvalues.

Proof. First, since q has non-negative coefficients, it is enough to show that q is hyperbolic with respect to a direction $a \in \mathbb{R}^n_{>0}$.

We use the following simple fact:

Fact 8.3. *Let $p \in \mathbb{R}^{n \times n}$ be a d-homogeneous polynomial hyperbolic with respect to $e \in \mathbb{R}^n$. Then, for any invertible matrix A, $r(z) = p(Az)$ is hyperbolic with respect to $A^{-1}e$.*

This simply follows from the fact that $r(x - tA^{-1}e) = p(A(x - tA^{-1}e)) = p(Ax - te)$.

Consider the polynomial $p = x_1^2 - x_2^2 - \cdots - x_n^2$. As we discussed in the last lecture this polynomial is hyperbolic with respect to $(1, 0, \dots, 0)$. Let $Q = \sum_{i=1}^n \lambda_i v_i v_i^T$ with $\lambda_1 > 0$ (and the rest negative). Define A to be the matrix where the i-th row is $\sqrt{|\lambda_i|} v_i^T$. Then, $q(z) = p(Az)$. So, by the above fact, q is hyperbolic with respect to any $a = A^{-1}e$ where $e \in K_+^p$.

Let $a = v_1$. Note that since Q has non-negative entries by Perron-Frobenius theorem, $a \in \mathbb{R}^n_{\geq 0}$. Then, $Aa = (\sqrt{\lambda_1}, 0, \dots, 0)$ is in the hyperbolicity cone of p as desired. □

8.2 Simple Properties of Log Concave Polynomials

In this section we discuss easy to prove properties of log-concave polynomials.

Closure Properties. Since the class of negative semidefinite matrices are closed, the set of log-concave polynomials are also are closed under taking limits.

The following closure property is immediate and we leave it as an exercise:

Lemma 8.4. *Let $p \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]$ be a log-concave polynomial. Then for any affine transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ with $T(\mathbb{R}^m_{\geq 0}) \subseteq \mathbb{R}^n_{\geq 0}$, $g(T(y_1, \dots, y_m)) \in \mathbb{R}[y_1, \dots, y_m]$ has nonnegative coefficients and is log-concave.*

Consequently, these polynomials are closed under any non-negative external field $(\lambda_1, \dots, \lambda_n)$, specialization to non-negative real numbers, $z_i \leftarrow c$ where $c \geq 0$, and symmetrization.

Lemma 8.5. *Let $p \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]$ be a d-homogeneous polynomial (for $d \geq 2$) and let $Q = \nabla^2 p(a)$ for some $a \in \mathbb{R}^n_{\geq 0}$ such that $p(a) \neq 0$. Then, Q has at most one positive eigenvalue if and only if p is log-concave at a.*

Note that, indeed, Q will have exactly one positive eigenvalue because all entries of Q are non-negative.

Proof. Euler's identity states that for a homogeneous polynomial p of degree d,

$$\sum_{i=1}^n z_i \partial_{z_i} p = d \cdot p.$$

Using this fact on p and $\partial_j p$ we get that

$$Qa = (d-1)\nabla p(a) \tag{8.1}$$
$$a^T Qa = d(d-1)p(a). \tag{8.2}$$

Having this, we can write

$$\nabla^2 \log p(a) = \frac{p\cdot\nabla^2 p - \nabla p \nabla p^T}{p^2}\Big|_{z=a}$$
$$= d(d-1)\frac{a^TQa\cdot Q - \frac{d}{d-1}(Qa)(Qa)^T}{(a^TQa)^2} = d(d-1)\frac{Q}{a^TQa} - d^2\frac{(Qa)(Qa)^T}{(a^TQa)^2}. \tag{8.3}$$

If p is log-concave at a, then the above quantity is negative semi-definite. Since p has non-coefficients, and $a \geq 0$, $a^TQa \geq 0$. Therefore, since the second term in the RHS is a rank one matrix, by the following theorem Q has at most one positive eigenvalue.

Theorem 8.6 (Cauchy Interlacing Theorem). *For a symmetric matrix $A \in \mathbb{R}^{n\times n}$ and vector $v \in \mathbb{R}^n$, the eigenvalues of A interlace the eigenvalues of $A + vv^T$, i.e., if $\alpha_1, \dots, \alpha_n$ are eigenvalues of A and $\beta_1, \dots, \beta_n$ are eigenvalues of $A + vv^T$ then*

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \beta_n.$$

Conversely, suppose Q has at most one positive eigenvalue. Let $b \in \mathbb{R}^n$. Consider the $n \times 2$ matrix P with columns a and b. Then

$$P^TQP = \begin{bmatrix} a^TQa & a^TQb \\ b^TQa & b^TQb \end{bmatrix}$$

We show that $\det(P^TQP) \leq 0$. This implies that $a^TQa \cdot b^TQb - (a^TQb)^2 \leq 0$ implying that $\nabla^2 \log p(a) \preceq 0$ by (8.3).

First, if P has rank one, then so does P^TQP, meaning that $\det(P^TQP) = 0$ and we are done. Otherwise P has rank two. Since L is negative semi-definite in a linear space of dimension $n-1$, there is a vector $v \in \mathbb{R}^2$ such that $(Pv)^TQ(Pv) \leq 0$. But this implies that P^TQP has a non-positive eigenvalue. On the other hand, since all entries of P, Q are non-negative, P^TQP has a positive eigenvalue. Therefore, $\det(P^TQP) \leq 0$ as desired. □

Necessary Conditions. Next, we discuss a necessary condition for a polynomial to be log-concave.

Definition 8.7 (Indecomposable Polynomials). *A polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ is* indecomposable *if it cannot be written as $p_1 + p_2$, where p_1, p_2 are nonzero polynomials in* disjoint sets *of variables, and we say it is* decomposable *otherwise. For example, $z_1^2 + z_2 z_e$ is decomposable whereas $z_1z_2 + z_1z_3$ is not.*

Equivalently, if we form a graph with vertices $\{i | \partial_{z_i} p \neq 0\}$ and edges $\{(i,j) | \partial_{z_i}\partial_{z_j} p \neq 0\}$, then p is indecomposable if and only if this graph is connected.

It turns out that indecomposability is a necessary condition for log-concavity.

Lemma 8.8. *If $p \in \mathbb{R}_{\geq 0}[z_1, \dots z_n]$ is homogeneous of degree at least 2 and is and log-concave at **1**, then p is indecomposable.*

Proof. For the sake of contradiction suppose $p = g + h$ is decomposable where $g \in \mathbb{R}[z_i : 1 \leq i \leq k]$ and $h \in \mathbb{R}[z_i : k+1 \leq i \leq n]$. Since both g and h are restrictions of p obtained by setting some variables equal to zero, both g and h are log-concave. Then, at **1**, the Hessians of g and h each have precisely one positive eigenvalue by Lemma 8.5. However, the Hessian of p at this point is a block diagonal matrix with these two blocks,

$$\begin{bmatrix} \nabla^2 g(\mathbf{1}) & 0 \\ 0 & \nabla^2 h(\mathbf{1}) \end{bmatrix}$$

So, $\nabla^2 p(\mathbf{1})$ has exactly two positive eigenvalues, meaning that p is not log-concave by Lemma 8.5, a contradiction. □

8.3 Matroids

In the second part of the course we will talk about matroids quite a bit. Let us first formally define matroids. A *matroid* $M = ([n], \mathcal{I})$ is defined on a ground set of elements, say $[n] = \{1, \ldots, n\}$ and a family of *independent sets* $\mathcal{I} \subseteq 2^{[n]}$ that satisfies the following properties:

Downward Closed: If $A \in \mathcal{I}$ then for any $B \subseteq A$, we have $B \in \mathcal{I}$.

Exchange Property: If $A, B \in \mathcal{I}$ and $|A| > |B|$ then there is an element $i \in B \setminus A$ such that $A \cup \{i\} \in \mathcal{I}$.

It follows from the exchange property that all *maximal* independent sets of a matroid M have the same size called the *rank* of the matroid. Any maximal independent set of a matroid is called a *base*.

Closure Properties. Given a matroid $M = ([n], \mathcal{I})$ of rank r, it is closed under many operations.

Contraction: For an element $1 \leq i \leq n$, M/i is the matroid on elements $[n] \setminus \{i\}$ with independent sets:

$$\{I : i \notin I, I \cup \{i\} \in \mathcal{I}\}.$$

Deletion For an element $1 \leq i \leq n$, $M \setminus i$ is the matroid on elements $[n] \setminus \{i\}$ with independent sets:

$$\{I : i \notin I, I \in \mathcal{I}\}.$$

Truncation For an integer $1 \leq k \leq r$, the truncation of M to k, M_k is the matroid with elements $[n]$ and independent sets:

$$\{I : |I| \leq k, I \in \mathcal{I}\}.$$

Examples. Matroids were defined and studied by Whitney around one hundred years ago in order to generalize the notion of linear independence in vector spaces. It is not hard to see that for any set of vectors $v_1, \ldots, v_n$ over a field F we can define a matroid, where any sets $A \subseteq [n]$ is an independent set if the corresponding set of vectors are linearly independent. The notion of rank in this case is the same as the rank of the vector space defined by of $v_1, \ldots, v_n$. Such a matroid is called a *linear* matroid. Another famous example of matroids is the *graphic matroid*. Here, $[n]$ is the set of edges of a graph G and a set of edges form an independent set if they do not induce a cycle. It is not hard to see that graphic matroids are special cases of linear matroids.

Although most of the matroids that we know can be represented by a linear matroid, almost all matroids are not linear. Nonetheless, as we see in the following lecture one can define and study many geometric structures based on this abstract structures. That is why Rota calls matroids *combinatorial geometries*.

Negative Correlation. As we explained before, many of the matroids cannot be realized as the support of the real stable polynomials, such as the Fano matroid. There is a fundamental reason why generating polynomials of matroids are not always real stable. This is because a uniform distribution over the bases of a matroid is not necessarily *negatively correlated*. For example, the $S8$ matroid is represented over GF(2) with the columns of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Note that the rank of this matrix is 4; so $\text{rank}(S8) = 4$. It is not hard to see that $|B_1| = 28$, $|B_8| = 20$, $|B_{1,8}| = 12$ and $|B| = 48$, where by B_1 we mean the set of bases that have column 1 and B is the set of all bases. The matroid is not negative correlated because

$$28 \cdot 20 \not\geq 12 \cdot 48.$$

Our main motivation to study generalizations of real stable polynomials is to find a class of polynomials that inherit almost all (closure) properties of real stable polynomials and furthermore all matroids can be realized as such polynomials. The following theorem is the main theorem

Theorem 8.9. *The bases generating polynomial of any matroid, $\sum_{B \text{ base}} z^B$ is a log-concave polynomial.*

Note that we have already proved the above theorem for the special case of spanning trees, i.e., bases of graphic matroids. This is because in such a case the bases generating polynomial is real stable.

Before proving the above theorem in its full generality, let us first speculate about the special case of rank 2 matroids.

Theorem 8.10. *Let $M = ([n], \mathcal{I})$ be a rank 2 matroid. Then, the bases generating polynomial is log concave (and real stable).*

Proof. Let $q(z) = \sum_B z^B$ be the bases generating polynomial of M. Note that q is 2-homogeneous; so we can write $q = z^T Q z$ where for any i, j,

$$Q_{i,j} = \begin{cases} 1 & \text{if \{i,j\} is a base} \\ 0 & \text{otherwise.} \end{cases}$$

In particular Q is 0 on the diagonal.

Here is the main observation: We say i, j are *parallel* if $\{i, j\}$ is not independent. It turns out this relation is *transitive*. If $\{i, j\}, \{j, k\} \notin \mathcal{I}$, then $\{i, k\} \notin \mathcal{I}$. Otherwise, the sets $\{j\}, \{i, k\}$ do not satisfy the exchange property of M.

Having this in hand we observe that each parallel class of elements of M form a 0 submatrix in Q. Say we partition elements $[n]$ into parallel classes $P_1, \dots, P_m$ it follows that for any i, j, $Q_{i,j} = 0$ if i, j belong to the same parallel class and 1 otherwise. Therefore,

$$Q = J - \sum_{i=1}^{m} J_{P_m}$$

where J is the all-ones matrix, and for each P_i, J_{P_i} is the all-ones matrix in the rows and columns indexed by P_i. Now, observe that each J_{P_i} is a rank 1 PSD matrix. Therefore, $-\sum_i J_{P_i}$ is negative semidefinite plus a rank 1 update J. So, by Cauchy Interlacing Theorem Q has at most one positive eigenvalue. $\square$

Given the above theorem, a natural approach to prove Theorem 8.9 is to use induction. Consider the bases generating polynomial of M under contraction, $g_{M/n}$, and the bases generating polynomial of M under deletion, M/n. Say, by induction hypothesis we know these polynomials are log-concave for any n. We can write $g_M = z_n g_{M/n} + g_{M\smallsetminus n}$. The question is when can we say the sum of two log-concave polynomials is log-concave?

8.4 Interlacing

Let us first address the aforementioned question (in the previous paragraph) for real stable polynomials. Given two real stable polynomials p, q when can we say $p + q$ is real stable?

First, we start with the univariate case.

Definition 8.11 (Interlacing)**.** *We say a real rooted polynomial $r = (\alpha_0)\prod_{i=1}^{d-1}(t - \alpha_i)$ of degree $d - 1$* interlaces *a polynomial $p = (\beta_0)\prod_{i=1}^{d}(t - \beta_i)$ if*

$$\beta_1 \le \alpha_1 \le \beta_2 \le \cdots \le \alpha_{d-1} \le \beta_d.$$

We say two degree d real rooted polynomials p, q have a common interlacer *if there is a polynomial r that interlaces both of thm.*

It is known that if p, q have a common interlacer then $p + q$ is real rooted; in fact, in that case $\lambda p + (1 - \lambda)q$ is real rooted for any $0 < \lambda < 1$. This definition naturally generalizes to real stable polynomials. We say a real stable polynomial $r \in \mathbb{R}[z_1, \ldots, z_n]$ (of degree $d - 1$ interlaces a real stable polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$ of degree d if for any $a \in \mathbb{R}^n_{>0}$ and any $x \in \mathbb{R}^n$, $r(x + ta)$ interlaces $p(x + ta)$. Similarly, two real stable polynomials p, q have a common interlacer if there is r that interlaces both. Naturally, if p, q have a common interlacer then $p + q$ is real stable.

We remark that interlacing properties of real stable polynomials, as a technique, recently had several applications in constructing Ramanujan graphs and in proving Kaidons-Singer problem.

Having this in mind, the question is how can we know if two real stable polynomials have a common interlacer? For a polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$ and a vector $a \in \mathbb{R}^n$, we write

$$D_a p = \sum_{i=1}^{n} a_i \partial_{z_i} p.$$

It is not hard to show that for any real stable polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$ and any vector $a \in \mathbb{R}^n_{\ge 0}$, $D_a p$ interlaces p. Consequently, given two polynomials p, q if there are vectors $b, c \in \mathbb{R}^n_{\ge 0}$ such that $D_b p = D_c q \ne 0$, then $D_b p$ is a common interlacer of p, q and $p + q$ is real stable.

It turns out that the above fact generalizes to log-concave polynomials.

Lemma 8.12. *Let $p, q \in \mathbb{R}_{\ge 0}[z_1, \ldots, z_n]$ be two homogeneous polynomials such that $D_b p = D_c q \ne 0$ for some $b, c \in \mathbb{R}^n_{\ge 0}$. If p, q are log-concave, then $p + q$ is log-concave.*

Note that in the above theorem we have a weaker assumption (log-concavity as opposed to stability) and a weaker conclusion.

We will prove this theorem in the following section. In the rest of this section we use this theorem to prove Theorem 8.9.

Lemma 8.13. *Let $p \in \mathbb{R}[z_1, \ldots, z_n]$ be homogeneous of degree $d \geq 3$ and indecomposable with nonnegative coefficients. If $\partial_{z_i} p$ is log-concave on $\mathbb{R}^n_{\geq 0}$ for every $1 \leq i \leq n$, then so is $D_a p$ for every $a \in \mathbb{R}^n_{\geq 0}$.*

Proof. First, if $\partial_{z_i} p$ is identically zero for some i, then we can consider p as a polynomial in the other variables. Without loss of generality, we assume that $\partial_{z_i} p$ is nonzero for all i. Furthermore, since p is indecomposable the pairs (i, j) where $\partial_{z_i}\partial_{z_j} p \neq 0$ form a connected graph which has a spanning tree. So, we can relabel the variables $z_1, \ldots, z_n$ so that for every $2 \leq j \leq n$, there exists $i < j$ for which $\partial_{z_i}\partial_{z_j} p \neq 0$.

Now, fix $a \in \mathbb{R}^n_{>0}$. We will show that $D_a p$ is log-concave on $R^n_{\geq 0}$. We show by induction on k that for any $1 \leq k \leq n$, $\sum_{i=1}^k a_i \partial_{z_i} p$ is log-concave on $\mathbb{R}^n_{\geq 0}$. The case $k = 1$ follows by assumption. For $1 \leq k < n$, let b denote the truncation of a to its first k coordinates, $b = (a_1, \ldots, a_k, 0, \ldots, 0)$ and let c denote the vector $a_{k+1}\mathbf{1}_{k+1}$. By induction hypothesis and lemma's assumption both $D_b p$ and $D_c p = a_{k+1}\partial_{k+1} p$ are log-concave. Since

$$D_c D_b p = D_b D_c p = \sum_{i=1}^k a_i a_{k+1} \partial_{z_i} \partial_{z_{k+1}} p$$

Note that each coefficient is non-negative and the sum is non-zero because for some $1\imath \leq k$, $\partial_{z_i}\partial_{z_{k+1}} p \neq 0$. Therefore, by Lemma 8.12, $D_b p + D_c p = \sum_{i=1}^{k+1} a_i \partial_{z_i} p$ is log-concave on $\mathbb{R}^n_{\geq 0}$. But, for $k = n - 1$, this is exactly $D_a p$ as desired. Finally, Taking closures then shows that Da f is log-concave on $\mathbb{R}^n_{\geq 0}$ for □

Till now both Lemma 8.12 and Lemma 8.13 also holds for real stable polynomials, i.e., if all partial derivatives of p are real stable and p has non-negative coefficient then so is any directional derivative of p. But the following important lemma does not hold for real stable polynomial.

Lemma 8.14. *Let $p \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$ be a d homogeneous polynomial with $d \geq 3$ and $a \in \mathbb{R}^n_{\geq 0}$ such that $p(a) \neq 0$. Then, p is log-concave at a iff $D_a p$ is log-concave at a.*

Proof. First, observe that $D_a p$ is homogeneous of degree ≥ 2. It follows by Euler's identity that

$$\nabla^2 (D_a p)|_{z=a} = (d-2)\nabla^2 p|_{z=a}.$$

Now, by Lemma 8.5 p is log-concave at a iff $\nabla^2 p(a)$ has exactly one positive eigenvalue and, similarly, $D_a p$ is log-concave at a iff $\nabla^2 (D_a p)(a)$ has at most one positive eigenvalue. Thus lemma's statement follows from the above inequality. □

Theorem 8.15. *Let $p \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$ be a d-homogenous polynomial. If the following two conditions are satisfied, then p is log-concave over $\mathbb{R}^n_{\geq 0}$.*

i) For all $\kappa \in \mathbb{Z}^n_{\geq 0}$ and $\|\kappa\|_1 \leq d - 2$, $\partial^\kappa p$ is indecomposable, where $\partial^\kappa = \prod_{i=1}^n \partial_{z_i}^{\kappa_i}$.

ii) For all $\kappa \in \mathbb{Z}^n_{\geq 0}$ and $\|\kappa\|_1 = d - 2$, the quadratic polynomial $\partial^\alpha p$ is log-concave over $\mathbb{R}^n_{\geq 0}$.

Proof. We prove the theorem by induction. The base case $d = 2$ obviously holds. Suppose for all $1 \leq i \leq n$, $\partial_{z_i} p$ is log-concave. Since p is indecomposable, by Lemma 8.13 for any $a \in \mathbb{R}^n_{\geq 0}$, $D_a p$ is log-concave over $\mathbb{R}^n_{\geq 0}$. Therefore, by Lemma 8.14 p is log-concave over $\mathbb{R}^n_{\geq 0}$. □

As an immediate consequence, here we prove Theorem 8.9.

Proof. Theorem 8.9 Let $M = ([n], \mathcal{I})$ be matroid of rank r. First, since g_M is multilinear, the only vectors $\kappa \in \mathbb{Z}^n_{\geq 0}$ with $\|\kappa\|_1 = r-2$ that $\partial^\kappa p \neq 0$ are vectors $\kappa \in \{0,1\}^n$. So, we think of κ as a set $S \subseteq [n]$.

Second, observe that for any i, $\partial_{z_i} g_M = g_{M/i}$, i.e., differentiation is the same as contraction. Therefore, for such any $S \subseteq [n]$ of size $|S| = r-2$, $\partial^S g_M = g_{M/S}$ is a matroid of rank 2 and by Theorem 8.9 we know that $\partial^\kappa g_M$ is log-concave.

Using Theorem 8.17 we only need to check the indecomposability condition. For any $S \subseteq [n]$ of size $|S| \leq r-2$, we need to show $g_{M/S}$ is indecomposable. First, for any pair of elements i, j in M/S, $\partial_{z_i}\partial_{z_j} g_{M/S} \neq 0$ iff i, j are non-parallel. Second, since M/S has rank at least 2, it must have at least two parallel classes of elements. But this immediately implies that $g_{M/S}$ is indecomposable. The statement follows from Theorem 8.17. □

8.5 Interlacing for Log Concave Polynomials

In this section we prove Lemma 8.12. The assumption that $D_b p = D_c q \neq 0$ means that f and g have the same degree d. We proceed by induction on d. If $d = 1$, then $p + q$ is a linear form with nonnegative coefficients, which is automatically log-concave on $\mathbb{R}^n_{\geq 0}$. Now suppose $d \geq 2$ and fix $a \in \mathbb{R}^n_{>0}$. Note that since neither p nor q can be identically zero, we have $p(a) > 0$ and $q(a) > 0$. Let $Q_1 = \nabla^2 p(a)$ and $Q_2 = \nabla^2 q(a)$. Then $D_b p = D_c q$ implies that for each $i = 1, \dots, n$,

$$(Q_1 b)_i \;=\; (\partial_i D_b p)|_{z=a} \;=\; (\partial_i D_c q)|_{z=a} \;=\; (Q_2 c)_i,$$

showing that $Q_1 b = Q_2 c$. Since $D_b p$ has nonnegative coefficients and is not identically zero, we also have that $D_b p(a) \neq 0$, meaning that $Q_1 b \neq 0$. Since $b \in \mathbb{R}^n_{\geq 0}$, by an argument similar to Lemma 8.1, we obtain that Q_1 is negative semidefinite on $(Q_1 b)^\perp$. Similarly, $c \in \mathbb{R}^n_{\geq 0}$, by a similar argument Q_2 is negative semidefinite on $(Q_2 c)^\perp$. But, $(Q_1 b)^\perp = (Q_2 c)^\perp$. Therefore, the matrix $Q_1 + Q_2 = \nabla^2 (p+q)|_{z=a}$ is also negative semidefinite on this $(n-1)$-dimensional linear space, and it has exactly one positive eigenvalue. Finally, by Lemma 8.5, $p + q$ is log-concave at $z = a$. This completes the proof of Lemma 8.12.

8.6 Completely Log-Concave Polynomials

Definition 8.16. *A d-homogeneous polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ is log-concave if for any set of vectors $a_1, \dots, a_k \in \mathbb{R}^n_{\geq 0}$, $D_{a_1} \dots D_{a_k} p$ is non-negative and log-concave over $\mathbb{R}^n_{\geq 0}$.*

Note that a completely log-concave polynomial p has non-negative coefficients. It follows from the same assumption of Theorem 8.17 that the underlying polynomial p is log-concave.

Theorem 8.17. *Let $p \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]$ be a d-homogenous polynomial. If the following two conditions are satisfied, then p is completely log-concave.*

i) For all $\kappa \in \mathbb{Z}^n_{\geq 0}$ and $\|\kappa\|_1 \leq d-2$, $\partial^\kappa p$ is indecomposable, where $\partial^\kappa = \prod_{i=1}^n \partial_{z_i}^{\kappa_i}$.

ii) For all $\kappa \in \mathbb{Z}^n_{\geq 0}$ and $\|\kappa\|_1 = d-2$, the quadratic polynomial $\partial^\alpha p$ is log-concave over $\mathbb{R}^n_{\geq 0}$.

Note that both of the above conditions are necessary for p be ing completely log-concave. So, this theorem gives a characterization for homogeneous completely log-concave polynomials.

Proof. By taking closures, we can assume that $a_1, \dots, a_k \in \mathbb{R}^n_{>0}$. First, if $k \geq d-1$ then either $D_{a_1} \dots D_{a_k} p$ is either identically zero or a linear polynomial with non-negative coefficients in which case it is (completely) log-concave. In Theorem 8.17 we proved the case $k = 0$, i.e., we showed that f is log-concave.

Now, suppose $1 \leq k \leq d-2$. Let $q = D_{a_1} \ldots D_{a_{k-1}} p$. Note that q has degree at least 3. By induction hypothesis (on k), $\partial_j p$ is completely log concave for any j. Hence,

$$D_{a_1} \ldots D_{a_{k-1}} \partial_{z_j} p = \partial_{z_j} D_{a_1} \ldots D_{a_{k-1}} p = \partial_{z_j} q$$

is log-concave on $\mathbb{R}^n_{\geq 0}$. Furthermore, since $a_1, \ldots, a_{k-1} > 0$, q is indecomposable. Since q has degree at least 3 by Lemma 8.13,

$$D_{a_k} q = D_{a_1} \ldots D_{a_k} p$$

is log-concave on $\mathbb{R}^n_{\geq 0}$ as desired. □