THE POLYNOMIAL PARADIGM IN ALGORITHMS. SOLUTIONS #1

1. Problem 1

Claim: For any n, k the elementary symmetric polynomial is real stable, where

$$e_k(z_1 ... z_n) = \sum_{S \in \binom{n}{k}} z^S$$

Proof. Linear functions are real stable if they have positive coefficients. Therefore, $(x + z_i)$ is real stable for all i. By closure properties,

$$p = \prod_{i=1}^{n} (x + z_i)$$

is real stable. Then specialize $x = 0$ for the polynomial $\frac{\partial^{n-k}}{\partial x^{n-k}} p$. This yields e_k (after scaling) since it preserves only the terms in p which chose exactly k terms in the product that do not include x. ☐

2. Problem 2

Claim: Let $p \in \mathbb{R}[z_1 ... z_n]$ be a homogeneous real stable polynomial. Then either all (non-zero) coefficients are positive or all negative.

Proof. Proof by induction on d, the degree of the monomials. We showed this in class for $d = 1$. So, assume it holds for some $d = k$ and we will show it for any homogeneous polynomial p of degree $d = k + 1$. Assume by way of contradiction that the monomials in p are not all the same sign. Then, first suppose that there exists two monomials of differing sign that share at least one variable z_i. Then the polynomial $\frac{\partial}{\partial z_i} p$ would contradict the induction hypothesis.

Otherwise, let P be the set of z_i that only appear in positive monomials and N the set that appears only in negative terms; by assumption it must be that $P \cap N = \varnothing$. Let P_S be the sum of coefficients of the positive monomials, and let N_S be the sum of coefficients of the negative monomials. Then setting $z_i = i$ for all $z_i \in P$ and $z_i = (\frac{P_S}{N_S})^{1/(k+1)} i$ elsewhere gives a 0 with all positive imaginary parts: contradiction. ☐

3. Problem 3

Claim: Let $p \in \mathbb{R}[z_1, z_2] = a + bz_1 + cz_2 + dz_1z_2$ be a real stable polynomial. Prove that $bc \geq ad$.

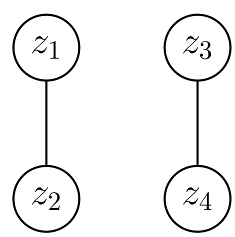
Proof. If $d = 0$ we are done since the polynomial is degree 1 so b, c are the same sign. Otherwise we know that $p(t - c/d, t - b/d)$ is real rooted. But this polynomial is equal to $a - \frac{bc}{d} + dt^2$. Therefore the discriminant gives $0 \geq 4(a - \frac{bc}{d})d$, or equivalently $bc \geq ad$ proving the claim. □

4. Problem 4

Claim: For some choice of G and k, the following polynomial is not real stable:

$$\sum_{M:|M|=k} \prod_{i \text{ sat in } M} z_i$$

Proof. Look at matchings with two nodes in the following graph:



This generates the polynomial $z_1z_2 + z_3z_4$. Yet it is not real stable. Here is a root in the upper half of the complex plane: set $z_1 = z_2 = 1 + i$ and $z_3 = z_4 = -1 + i$. □