

## Problem Set 1

*Deadline: October 27th in Canvas*

- 1) Use coupling to show that for any markov chain  $\tau(\epsilon) \leq O(\tau_{mix} \log(1/\epsilon))$ .
- 2) We say a sequence  $Y_0, Y_1, \dots$  is a *martingale* with respect to another sequence  $X_0, X_1, \dots$  if for all  $t$ ,

$$\mathbb{E}[Y_{t+1} | X_0, \dots, X_t] = Y_t.$$

Consider the unbiased random walk on the integer line. That is we start at 0, and at each time step we stay with probability  $\theta$  and we move left/right uniformly at random otherwise. Let  $X_t$  be the location at time  $t$ . For example, observe that for  $Y_t = 2 \cdot X_t$  is a martingale with respect to  $X_t$ .

Note that for any martingale, and any time  $t$  we obviously have  $\mathbb{E}[Y_t] = \mathbb{E}[Y_{t-1}] = \dots = \mathbb{E}[Y_0]$ . Let  $T$  be a stopping time. The optional sampling theorem says that if  $\mathbb{E}[T]$  is bounded, and, for all  $t < T$ ,  $\mathbb{E}[|Y_{t+1} - Y_t| | X_0, \dots, X_t] \leq c$  with probability 1, then  $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$ .

Construct a martingale and use optional stopping theorem to show that a random walk started at 0 hits  $+b$  or  $-b$  in  $\frac{b^2}{1-\theta}$  steps in expectation.

- 3) Consider the simple random walk on the hypercube  $\{0, 1\}^n$  where at each vertex we stay with probability  $\frac{1}{n+1}$  and we move to a uniformly random neighbor otherwise. Use coupling to prove that
- $\tau_{mix} \leq \frac{1}{2}n \ln n + O(n)$ .
  - Optional:**  $\tau_{mix} \leq \frac{1}{4}n \ln n + O(n)$ .
- 4) Recall that in the Ising model, the probability of a configuration  $\sigma$  is proportional to  $\prod_{i \sim j} e^{\beta \sigma_i \sigma_j}$ . Consider the Heat-Bath chain that we discussed in class; it also goes by the name Glauber dynamics. That is we sample  $\sigma_i$  from the stationary distribution conditioned on  $\sigma_j$  for all neighbors  $i$ . Suppose we run this chain on a  $\sqrt{n} \times \sqrt{n}$  torus. That is we assume that vertices in the last column are adjacent to vertices in the first column and similarly the vertices in the last row are adjacent to vertices in the first row. Use Path Coupling to show that there is a universal constant  $c > 0$  such that for all  $0 < \beta < c$ , the chain mixes in time  $O(n \log n)$ .