Counting and Sampling **Fall 2017**

Lecture 15: Barvinok's Method: Approximating the Matching Polynomial

Lecturer: Shayan Oveis Gharan *November 17th*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

Recall that for a graph G the (positive) matching polynomial is defined as follows:

$$\mu_G^+(x) = \sum_{k=0}^{n/2} m_k x^k,$$

where m_k is the number of matchings of G of size k. In this lecture we prove the following theorem of Patel and Regts [PR17]

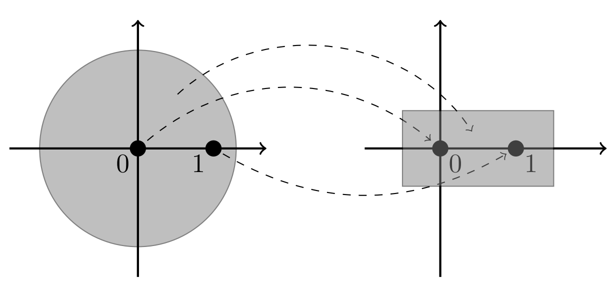
Theorem 15.1. *For any graph G of maximum degree Δ and any t that does not lie on negative real axis we can give a $1+\epsilon$ multiplicative approximation of $\mu_G^+(t)$ in time polynomial in $n, 1/\epsilon$ and exponential in t, Δ.*

Note that unlike the proof in lecture 12 which only approximated the matching polynomial on positive reals here we can approximate it almost anywhere.

We will use an extension of the Barvinok's technique that we discussed last time. Perhaps, the first idea that comes to mind is to argue that the matching polynomial has no roots in a ball of radius $\beta > 1$ around the origin. Furthermore, we know that $\mu_G^+(0) = m_0 = 1$. This idea fails as we will show later that μ_G^+ has a root as small as $-1/\Omega(\Delta)$. Instead we will use the following facts that we will prove later: (i) μ_G^+ is real rooted, (ii) It has no positive roots, (iii) the largest root of μ_G^+ is no larger than $-1/4(\Delta-1)$.

So, for a constant Δ, all roots of μ_G^+ are far from the line that connects 0 to 1. The idea of Barvinok is that whenever there is a curve from 0 to 1 that is "far" from all roots of the polynomial we should be able to estimate the polynomial at 1 using its logarithmically many derivatives at 0.

Suppose $g(z)$ is a polynomial that has no roots close to the strip from 0 to 1, i.e., for all points z where $-\delta \leq \Re z \leq 1+\delta$ and $|\Im z| \leq \delta$, $g(z) = 0$. Here is the idea: We construct a function ϕ such that $\phi(0) = 0$, $\phi(1) = 1$ such that it maps the disc $|z| \leq \beta$ into the strip $-\delta \leq \Re z \leq 1+\delta$, $|\Im z| \leq \delta$.



Then, the polynomial $g(\phi(z))$ has no roots in the ball of radius β around the origin. So we can use the polynomial approximation lemma from the last lecture to approximate $g(\phi(z))$ by the taylor expansion around 0 of degree $O(\log degg + \log \deg \phi + \log 1/\epsilon)$ to obtain a multiplicative $1+\epsilon$ approximation of $g(\phi(z))$.

Let $h(z) = g(\phi(z))$. Recall that to compute the Taylor polynomial of degree n of $\ln h(z)$ at $z = 0$ it suffices to compute the Taylor polynomial of degree n of $h(z)$ at $z = 0$. In other words, we just need the bottom n

coefficients of the polynomial $h(z)$. But this can be computed from the bottom n coefficients of $g(.)$ and the bottom n coefficients of $\phi(.)$.

Now, all we need is to construct the polynomial $\phi(.)$.

Lemma 15.2 ([Bar17]). *For $0 < \delta < 1$ define*

$$\phi(z) = \phi_\delta(z) = \frac{1}{\sigma}\sum_{k=1}^{N}\frac{(\alpha z)^k}{k},$$

where $\sigma = \sum_{k=1}^{N}\frac{\alpha^k}{k}$ is the normalizing constant to make sure $\phi(1) = 1$, $N = O(e^{1/\delta})$ is the degree and $\alpha = 1 - e^{-1/\delta}$. Then, for all $|z| \leq \frac{1-e^{-1-1/\delta}}{1-e^{-1/\delta}} =: \beta$ we have

$$\begin{aligned} -\delta \leq \quad \Re\phi(z) \quad &\leq 1 + 2\delta \\ |\Im\phi(z)| \quad &\leq 2\delta. \end{aligned}$$

Proof. Note that clearly $\phi(0) = 0$. We work with the function

$$f_\delta(z) = \delta \ln \frac{1}{1 - \alpha z}$$

for $|z| \leq 1$. There is a technical fact that the complex log is not well defined; in particular we can have $\ln 1 = 2k\pi i$ for any integer k. So, here we choose the "branch" and let $\ln 1 = 0$. Therefore, $f(0) = 0$ and

$$f(1) = \delta \ln \frac{1}{1 - (1 - e^{-1/\delta})} = \delta \ln e^{1/\delta} = 1.$$

Now the following facts are easy to check:

$$-\delta \ln 2\Re f_\delta(z) \leq 1 + \delta,$$

and that $|\Im f_\delta(z)| \leq \pi\rho/2$. The actual constants do not matter in above inequalities. The main fact is that because we choose a branch of the log where $\ln 1 = 0$, we can write every complex number as a $re^{2\alpha\pi i}$ for some $0 \leq \alpha \leq 1$. Therefore, for every complex number z, $\Im \ln z < 2\pi$. This implies that $\Im \ln \frac{1}{1-\alpha z} \leq 2\pi$ over the whole complex plane, and δ just scales this imaginary value down.

The rest of the proof is similar to the arguments we had in the last lecture. We approximate f_δ by its taylor approximation □

15.1 Properties of the Matching Polynomial

For a graph G let

$$\mu_G(x) = \sum_{k=0}(-1)^k m_k x^{n-2k}.$$

The following is the main theorem that we prove in this section:

Theorem 15.3. *For any graph G with maximum degree Δ the largest root of $\mu_G(x)$ is at most $2\sqrt{\Delta - 1}$.*

Here is the general plan to prove the theorem. First we prove the theorem for trees with maximum degree Δ and then we show that the trees are indeed the worst case for the above theorem. So, let us start by studying the matching polynomial of a tree.

Lemma 15.4. *For any tree T, the matching polynomial of T is equal to characteristic polynomial of the adjacency matrix of T, i.e.,*

$$\mu_T(x) = \det(xI - A),$$

where A is the adjacency matrix of T.

Proof. To prove the theorem it is enough to show that for every k the coefficient of x^{n-2k} in the two polynomials are the same. Let us prove this fact only for the constant term, x^0 and we will see that the proof can be naturally extended to all k's. Note that the coefficient of x^0 in the $\mu_T(x)$ is the number of perfect matchings of G. That coefficient in $\det(xI - A)$ is exactly the determinant of A, i.e.,

$$\sum_\sigma \text{sgn}(\sigma) \prod_i A_{i,\sigma_i}.$$

We claim that $\prod_i A_{i,\sigma_i}$ is nonzero if and only if σ corresponds to a perfect matching in T. Firstly observe that this product is nonzero if for all i, $A_{i,\sigma_i} = 1$. Now, recall that any permutation is a union of cycles, say $i, \sigma_i, \sigma_{\sigma_i}, \text{etc}, i$. But because T is a tree it does not have any cycles. So, the only feasible cycles in σ are cycles of length 2, i.e., we must have $\sigma_{\sigma_i} = i$ for all i. In other words σ corresponds to a perfect matching of T. It follows that the sign of all these permutations is exactly $(-1)^{n/2}$. Therefore, the constant coefficient of $\mu_T(x)$ is the same as the constant coefficient of $\det(xI - A)$.

For other coefficients the proof is almost similar. Just note that the coefficient of x^{n-2k} of $\det(xI - A)$ is the sum of determinant of all principal $2k \times 2k$ minors of A. By a similar argument, each such determinant is equal to the number of perfect matching in the corresponding induced subgraph of T with $2k$ vertices. ☐

Now, we are ready to upper bound the largest root of $\mu_T(x)$ for a tree T.

Theorem 15.5. *Let T be a tree with maximum degree at Δ. The largest root of $\mu_T(x)$ is at most $2\sqrt{\Delta - 1}$.*

Proof. By Lemma 15.4 it is enough to upper bound the largest root of $\det(xI - A)$ where A is the adjacency matrix of T. But the roots of $\det(xI - A)$ are correspond to the eigenvalues of the adjacency matrix. So, we just need to upper bound the largest eigenvalue of A.

We use the trace method. Recall that for any symmetric matrix M, $\text{Tr}(M) = \sum_i \lambda_i$ where λ_i are eigenvalues of M. Therefore, for any k, $\text{Tr}(M^k) = \sum_i \lambda_i^k$. It follows that $\text{Tr}(M^k)^{1/k}$ is always an upper bound on the largest eigenvalue of M. Furthermore, it is not hard to see that as $k \to \infty$, $\text{Tr}(M^k) \to \lambda_{\max}$. Therefore, to prove the claim it is enough to show that

$$\lim_{k\to\infty} \text{Tr}(A^k) \leq (2\sqrt{\Delta - 1})^k = 2^k(\Delta - 1)^{k/2}.$$

Observe that for any vertex u, $A^k_{u,u}$ is the number of closed walks of length k that starts at u. To prove the above inequality we show that for any vertex u,

$$\lim_{k\to\infty} A^k_{u,u} \leq 2^k(\Delta - 1)^{k/2}. \tag{15.1}$$

Note that this is enough we would get

$$\lim_{k\to\infty} \text{Tr}(A^k)^{1/k} \leq \lim_{k\to\infty} n^{1/k}(2\sqrt{\Delta - 1}) = 2\sqrt{\Delta - 1}.$$

So, it remains to prove (15.1). Let us make the tree rooted at u. We can map any closed walk starting at u of length k with a sequence of $\uparrow, \downarrow$ of length k where if we have $\uparrow$ in position i it means that in the i-th step we move towards the root and otherwise it means we move against the root. Note that in order to have a closed walk any such sequence would have exactly $k/2$, $\uparrow$ symbols and exactly $k/2$, $\downarrow$ symbols. Every time that we go towards the root there is a unique edge that we can choose and every time that we go down there are at most $\Delta - 1$ edges that we can choose. Therefore, the number of closed walks of length k starting at the root is at most

$$1^{k/2}(\Delta - 1)^{k/2}\binom{k}{k/2},$$

where the last term corresponds to all possible ways to construct a sequence of length k with exactly $k/2$, $\uparrow$'s and exactly $k/2$, $\downarrow$'s. Equation (15.1) follows from the fact that $\binom{k}{k/2} \leq 2^k$. $\square$

The following fact can be proven similar to the facts on $\mu^+_G(x)$ that we proved in lecture 12.

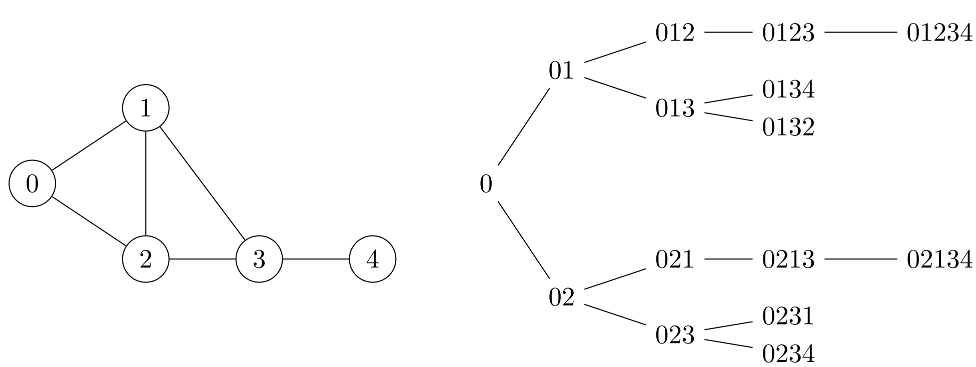
Fact 15.6. *For any pair of disjoint graphs G, H*

$$\mu_{G\cup H}(x) = \mu_G(x) \cdot \mu_H(x).$$

For any graph G and any vertex u,

$$\mu_G(x) = x\mu_{G-u}(x) - \sum_{v\sim u} \mu_{G-u-v}(x).$$

For a graph G and a vertex u, the *path-tree* of G with respect to u, $T = T_G(u)$ is defined as follows: For every path in G that starts at u, T has a node and two paths are adjacent if their length differs by 1 and one is a prefix of another. See the following figure for an example:



We prove the following theorem due to Godsil and Gutman.

Theorem 15.7. *Let G be a graph and u be a vertex of G. Also, let $T = T(G, u)$ be the path tree of G with respect to u. Then*

$$\frac{\mu_G(x)}{\mu_{G-u}(x)} = \frac{\mu_T(x)}{\mu_{T-u}(x)}.$$

Furthermore, $\mu_G(x)$ divides $\mu_T(x)$.

Note that this would directly imply Theorem 15.3 because if the maximum degree of G is at most Δ so is the maximum degree of $T(G, u)$. The above theorem implies that the root of $\mu_G(x)$ are a subset of the roots of $\mu_{T(G,u)}(x)$. And, by Theorem 15.5 the largest root of $\mu_{T(G,u)}(x)$ is at most $2\sqrt{\Delta - 1}$.

Proof. Firstly, observe that the theorem obviously holds when G is a tree because the path-tree of a tree is itself. So, suppose (inductively) that the theorem holds for all subgraph so of G. Let us write $H = G - u$.

Then, using the first part of Fact 15.6 we have

$$\begin{aligned}\frac{\mu_G(x)}{\mu_H(x)} &= \frac{x\mu_{G-u}(x) - \sum_{v\sim u}\mu_{G-u-v}(x)}{\mu_H(x)} \\ &= x - \sum_{v\sim u}\frac{\mu_{H-v}(x)}{\mu_H(x)} = x - \sum_{v\sim u}\frac{\mu_{T(H,v)-v}(x)}{\mu_{T(H,v)}(x)}. \end{aligned} \tag{15.2}$$

The last equality simply follows by the induction hypothesis. Here is the main observation: $T(H,v) = T(G-u,v)$ is isomorphic to the component of $T(G,u)-u$ which contains the point u,v. Therefore,

$$\frac{\mu_{T(H,v)-v}(v)}{\mu_{T(H,v)}(x)} = \frac{\mu_{T(G,u)-u-v}(x)}{\mu_{T(G,u)-u}(x)}.$$

Note that the rest of the connected components of $T(G,u)-u$ are also connected components of $T(G,u)-u-v$ so they will be cancelled out in the RHS (see second part of Fact 15.6).

So, we can rewrite the RHS of (15.2) as follows:

$$\begin{aligned} x - \sum_{v\sim u}\frac{\mu_{T(H,v)-v}(x)}{\mu_{T(H,v)}(x)} &= x - \sum_{v\sim u}\frac{\mu_{T(G,u)-u-v}(x)}{\mu_{T(G,u)-u}(x)} \\ &= \frac{x\mu_{T(G,u)-u}(x) - \sum_{v\sim u}\mu_{T(G,u)-u-v}(x)}{\mu_{T(G,u)-u}(x)} = \frac{\mu_{T(G,u)}(x)}{\mu_{T(G,u)-u}(x)}. \end{aligned}$$

This proves the first part of the theorem.

Now we prove the second part of the theorem. Firstly, by the first part we can write

$$\mu_T(x) = \mu_G(x)\cdot\frac{\mu_{T-u}(x)}{\mu_{G-u}(x)}.$$

To prove the second part we need to show that the ratio $\mu_{T-u}(x)$ is divisible by $\mu_{G-u}(x)$. Firstly, note that $\mu_{T-u}(x) = \mu_{T(G,u)-u}(x)$ is divisible by $\mu_{T(G-u,v)}(x)$. This is because the latter is isomorphic to one of the connected components of the former. Secondly, by induction $\mu_{T(G-u,v)}(x)$ is divisible by $\mu_{G-u}(x)$. Putting these together, we get $\mu_{T-u}(x)$ is divisible by $\mu_{G-u}(x)$. $\square$

References

[Bar17] A. Barvinok. Approximating permanents of hafnians, 2017. 15-2

[PR17] Viresh Patel and Guus Regts. Deterministic polynomial-time approximation algorithms for partition functions and graph polynomials. *Electronic Notes in Discrete Mathematics*, 61(Supplement C):971 – 977, 2017. The European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB). 15-1