$$=h'_{t_i}(\chi(t_i)|\chi(t_i) \cap \chi(\mathsf{parent}(t_i))) + (E_{T_{i-1}} \circ \varphi)(h)$$
(by Eq.(57))
$$=h(\varphi(\chi(t_i))|\varphi(\chi(t_i) \cap \chi(\mathsf{parent}(t_i)))) + (E_{T_{i-1}} \circ \varphi)(h)$$
(Lemma C.3 (1))
$$=(E_{T_i} \circ \varphi)(h) \quad \text{(Definition of } E_T\text{)}$$

This completes the inductive proof.

By setting $i = m$ (the number of nodes in T) in Eq.(55) and (56) we derive Eq.(53) and (54). The proof of the theorem follows from:

$$\begin{aligned}
\log|\mathsf{hom}(Q_1, \mathcal{D})| &= \log|P| \\
&= h(\mathsf{vars}(Q_1)) \leq (E_T \circ \varphi)(h) && \text{(by Eq. (51))} \\
&= E_T(h') && \text{(by Eq.(54))} \\
&= h'(\mathsf{vars}(Q_2)) && \text{(by Eq.(53))} \\
&\leq \log|P'| && \text{(Since } P' \text{ is the support of } h'\text{)} \\
&\leq \log|\mathsf{hom}(Q_2, \mathcal{D})| && \text{(By Eq (52))}
\end{aligned}$$

□

D PROOF OF THEOREM 3.3 AND 3.6

In Th.4.4 we proved that, when Q_2 is acyclic and Eq.(13) fails, then $Q_1 \not\preceq Q_2$. We prove here a variation of that result: when Q_2 is chordal and Eq.(13) fails on a normal entropic function, then $Q_1 \not\preceq Q_2$. Recall that a junction tree is a special tree decomposition.

Lemma D.1. *Let Q_2 be chordal and admit a simple junction tree T, and let E_T be its linear expression, Eq.(12). If there exists a normal entropic function h (i.e. with a non-negative I-measure) such that:*

$$h(\mathsf{vars}(Q_1)) > \max_{\varphi \in \mathsf{hom}(Q_2, Q_1)} (E_T \circ \varphi)(h) \tag{60}$$

then there exists a database instance $\mathcal{D}$ such that $|\mathsf{hom}(Q_1, \mathcal{D})| > |\mathsf{hom}(Q_2, \mathcal{D})|$.

We first show how to use the lemma and the essentially-Shannon inequalities in Th 3.9 to prove Theorems 3.3 and 3.6. Assume Q_2 is chordal and has a simple junction tree T. We prove: $Q_1 \preceq Q_2$ iff Eq.(13) holds. It suffices to prove that Eq.(13) is necessary, because sufficiency follows from Th. 4.2. Suppose Eq.(13) fails. Then there exists an entropic function h such that (60) holds where T in (60) is a simple junction tree of Q_2. Since T is simple, the conditional linear expressions on the right-hand-side of (60) are also simple. By Th 3.9, there exists a *normal* entropic function h such that (60) holds. Then, by Lemma D.1, $Q_1 \not\preceq Q_2$. This proves that Eq.(13) is necessary and sufficient for containment. Furthermore, Eq.(13) is decidable, since it is an essentially-Shannon inequality, and this completes the proof of Theorems 3.3. The proof of Theorem 3.6 follows immediately from the fact that the set of normal entropic functions $\mathcal{N}_n$ is the cone generated by the entropies of normal relations, and the set of modular functions $\mathcal{M}_n$ is the cone generated by the entropies of product relations.

It remains to prove Lemma D.1; the lemma generalizes Theorem 3.2 of [22] to arbitrary vocabularies (beyond graphs). To prove the theorem, we will update the proof of Theorem 4.4, where we used acyclicity of Q_2: more precisely we need to re-prove the locality property, Eq.(22). We repeat it here:

$$\mathsf{hom}_{\varphi|_{\chi(t)}}(Q_t, \mathcal{D}) \subseteq \Pi_{\varphi|_{\chi(t)}}(P)$$

We start by observing that this property fails in general.

Example D.2. Let $Q_1 = R(X_1, X_2), S(X_2, X_3), T(X_3, X_1)$ and $Q_2 = R(Y_1, Y_2), S(Y_2, Y_3), T(Y_3, Y_1)$ (they are identical). Consider the parity function in Example 3.8; more precisely, this is the entropy of the relation $P = \{(X_1, X_2, X_3) \mid X_1, X_2, X_3 \in \{0, 1\}, X_1 \oplus X_2 \oplus X_3 = 0\}$, which we show here for clarity:

$P =$

0	0	0
0	1	1
1	0	1
1	1	0

Recall that the entropy of P is not a normal entropic function (Sec. 6). This relation is perfectly uniform (in fact it is a group characterization). Computing $\mathcal{D} = \Pi_{Q_1}(P)$ we obtain $R^D = S^D = T^D = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Q_2 is a clique, with a bag $Q_t = Q_2$, and $\mathsf{hom}(Q_t, \mathcal{D})$ contains one extra triangle, $(1, 1, 1)$, which is in no single row of P.

The example shows that we need to use in a critical way the fact that the counterexample h is a normal entropic function, $h \in \mathcal{N}_n$. To use this fact, we will describe a class of relations whose entropic functions generate precisely the cone $\mathcal{N}_n$, and prove that these are precisely the normal relations (Def. 3.5).

Consider the normal entropic function h given by Lemma D.1. We can assume w.l.o.g. that h is a sum of step functions[13], $h = \sum_i h_{W_i}$, where each h_{W_i} is a step function (not necessarily distinct). Recall from Section 3.3 that P_{W_i} is the 2-tuple relation whose entropy is h_{W_i}; to reduce clutter we denote here P_{W_i} by P_i. Then h is the entropy of their domain-product (Def B.1), $P = P_1 \otimes P_2 \otimes \cdots \otimes P_m$. One can check that P is totally uniform (it is even a group realization). We now prove the locality property, Eq.(22), using the fact that P is a domain product, which allows us to rewrite Eq.(22) as:

$$\mathsf{hom}_{\varphi|_{\chi(t)}}(Q_t, \mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_m) \subseteq \Pi_{\varphi|_{\chi(t)}}(P_1 \otimes \cdots \otimes P_m)$$

It suffices prove that $\mathsf{hom}_{\varphi|_{\chi(t)}}(Q_t, \mathcal{D}_i) \subseteq \Pi_{\varphi|_{\chi(t)}}(P_i)$ for each i. Recall that P_i has two tuples, $P_i = \{f_1, f_2\}$, where $f_1 = (1, 1, \ldots, 1)$ and f_2 has values 1 on positions $\in W$ and values 2 on positions $\notin W$, for some set of attributes W. Fix a tuple $g \in \mathsf{hom}_{\varphi|_{\chi(t)}}(Q_t, \mathcal{D}_i)$; we must prove that either $g \in \Pi_{\varphi|_{\chi(t)}}(f_1)$ or $g \in \Pi_{\varphi|_{\chi(t)}}(f_2)$. If g maps every variable in $\mathsf{vars}(Q_t)$ to 1, then the first condition holds, so assume that g maps some variable $Y \in \mathsf{vars}(Q_t)$ to 2; in particular, $\varphi(Y) \notin W$. We must prove that, for every variable Y', if $\varphi(Y') \notin W$ then $g(Y') = 2$. Here we use the fact that Q_2 is chordal, hence Q_t is a clique, thanks to Fact A.2. Therefore, there exists $B \in \mathsf{atoms}(Q_t)$ that contains both Y and Y'. Since g is a homomorphism, it maps B to some tuple in $\Pi_{\varphi(\mathsf{vars}(B))}(P)$; since both $\varphi(Y), \varphi(Y') \notin W$, this tuple must have the value 2 on both positions (they can be identical: $\varphi(Y) = \varphi(Y')$). It follows that all variables Y' s.t. $\varphi(Y') \notin W$ are mapped to 2, proving that $g \in \Pi_{\varphi|_{\chi(t)}}(f_2)$. This proves the local property, Eq.(22). The rest of the proof of Theorem 4.4 remains unchanged, and this completes the proof of Lemma D.1.

[13] Suppose the contrary, that the inequality holds for all functions h that are sums of step functions. Then it holds for all linear combinations $\sum_W c_W h_W$ where $c_W \geq 0$ are integer coefficients. If an inequality holds for h, then it also holds for $\lambda \cdot h$ for any constant $\lambda > 0$; it follows that the inequality holds for all linear combinations $\sum_W c_W h_W$ where $c_W \geq 0$ are rationals. The topological closure of these expressions is $\mathcal{N}_n$, contradicting the fact that the inequality fails on some $h \in \mathcal{N}_n$.