BLACK SCHOLES
THE MARTINGALE APPROACH

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1. Introduction

This paper establishes the Black Scholes formula in the martingale, risk-neutral valuation framework. The intent is two-fold. One, to serve as an introduction to expectation pricing and two, to examine this framework in explicit mathematical detail. The reader is assumed to have fluent background in the mathematical theory of stochastic processes and calculus, but is not assumed to have background in finance. The relevant financial definitions are given in section 2.

Section 2 also establishes the main result (proposition 2.3) that links equivalent martingale measures (EMMs) to expectations. The key mathematical tool at work here is the martingale representation theorem, which guarantees the existence of a hedging strategy under an EMM. In section 3, we exploit this result by explicitly finding an EMM for geometric brownian motion (proposition 3.3). Here the key mathematical tool is Girsanov's theorem, which tells us how to convert to and from an EMM. Readers with financial background will recognize this conversion as substracting the market price of risk.

For general contingent claims we of course cannot reduce valuation to a simple formula. But the techniques used here indicate how we ought to proceed numerically. It is a two-step process. First, we back out an EMM from underlying market prices. Then we compute the value of the claim as its discounted expectation under the risk neutral measure. There is a potential obstacle to this procedure: an EMM might not exist. For this we appeal to a powerful guarantee called the fundamental theorem of asset pricing, which asserts the existence of an EMM if and only if the market is arbitrage-free.

2. Theory

Let S be a continuously traded asset and B be a risk free bond, often referred to as the **numeraire**. We assume throughout that the bond B is governed (under the **physical measure** P) by

$$dB_t = rB_t dt.$$

We say that r is the **spot rate of interest**, also called the **short rate**. If we hold X amount of S and Y amount of B over a period from $[0,t]$ then the **gains** of our portfolio over $[0,t]$ are given by

$$G_t = \int_0^t X_s dS_s + \int_0^t Y_s dB_s.$$

We call (X, Y) a **trading strategy**. The **value** of this trading strategy at time t is just a linear combination

$$V_t = X_t S_t + Y_t B_t.$$

We say that a strategy is **self-financing** if $V_t = V_0 + G_t$ for all t, and we define an **arbitrage opportunity** to be a self-financing trading strategy (X, Y) for which $V_0 = 0$, $V_t \geq 0$, and $\mathbb{E}(V_t) > 0$ for some $t > 0$.

For any process H we define $\tilde{H} = B^{-1}H$ and we say that $\tilde{H}$ has been **discounted** by the numeraire. It can be shown that if a trading strategy is self-financing then the discounted value process equals the initial value plus the gains in the discounted asset process. We will not need this result but we will make use of the converse:

Proposition 2.1. *Suppose that*

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t X_s d\tilde{S}_s.$$

Then the trading strategy (X, Y) is self-financing.

Proof. Observe that B is continuous with locally bounded variation and therefore its quadratic covariation with any other process vanishes. Combining this with integration by parts,

$$S_t = B_t \tilde{S}_t = \int_0^t \tilde{S}_s dB_s + \int_0^t B_s d\tilde{S}_s + [B, \tilde{S}]_s = \int_0^t \tilde{S}_s dB_s + \int_0^t B_s d\tilde{S}_s$$

And it follows that

$$\int_0^t X_s dS_s = \int_0^t B_s X_s d\tilde{S}_s + \int_0^t X_s \tilde{S}_s dB_s.$$

Applying integration by parts to the value process,

$$\begin{aligned}
V_t = B_t \tilde{V}_t &= \int_0^t B_s d\tilde{V}_s + \int_0^t \tilde{V}_s dB_s = \int_0^t B_s X_s d\tilde{S}_s + \int_0^t B^{-1} V_s dB_s \\
&= \int_0^t B_s X_s d\tilde{S}_s + \int_0^t B_s^{-1} X_s S_s dB_s + \int_0^t B_s^{-1} Y_s B_s dB_s \\
&= \int_0^t B_s X_s d\tilde{S}_s + \int_0^t X_s \tilde{S}_s dB_s + \int_0^t Y_s dB_s = \int_0^t X_s dS_s + \int_0^t Y_s dB_s = G_t.
\end{aligned}$$

Therefore (X, Y) is self-financing. $\square$

Suppose there is some measure $Q \sim P$ and a Q square-integrable martingale M such that

$$dS_t = rS_t dt + dM_t.$$

Then we say that Q is a **spot martingale measure** (for S). Likewise, if there is some $Q \sim P$ such that $\tilde{S}$ is a Q martingale then we say that Q is an **equivalent martingale measure** (for S). These definitions are roughly equivalent and we will show in the next proposition that the first implies the second. The converse is morally true, but we don't need it and it requires additionally the square-integrability of $\tilde{S}$.

Proposition 2.2. *If $Q \sim P$ is a spot martingale measure then it is an equivalent martingale measure.*

Proof. By classical calculus,

$$dB_t^{-1} = -rB_t^{-1}dt.$$

Integrating by parts with this result gives us

$$\tilde{S}_t = B_t^{-1}S_t = \int_0^t B_s^{-1}dS_s + \int_0^t S_s dB_s^{-1}.$$

From our hypothesis that Q is a spot martingale measure we therefore have

$$\tilde{S}_t = \int_0^t r\tilde{S}_s ds + \int_0^t B_s^{-1}dM_s - \int_0^t r\tilde{S}_s ds = \int_0^t B_s^{-1}dM_s.$$

Because M is square-integrable it follows that $\tilde{S}_t$ is a Q (square-integrable) martingale, i.e. Q is an equivalent martingale measure. $\square$

Let C be a **terminal claim** on S at time T. That is $C : [0,T] \to \mathbb{R}$ and $C_T = f(S_T)$. We say that a self-financing strategy **replicates**, or **hedges**, a terminal claim if $V_T = C_T$. If we can replicate the value of a claim at time T then, to prevent arbitrage, the value of the claim at an earlier time must equal the value of the replicating portfolio. We will now demonstrate such a hedging strategy. The following statement forms the basis for martingale pricing.

Proposition 2.3. *Suppose S admits an equivalent martingale measure Q. In the absence of arbitrage opportunities,*

$$C_t = B_t\mathbb{E}_Q\{B_T^{-1}C_T|\mathcal{F}_t\}.$$

Proof. Define $M_t = B_t\mathbb{E}_Q\{B_T^{-1}C_T|\mathcal{F}_t\}$ and observe that $B^{-1}M$ is a Q martingale. If $\tilde{W}$ denotes Q Brownian motion then by martingale representation,

$$\tilde{M}_t = M_0 + \int_0^t H_s d\tilde{W}_s.$$

By hypothesis $\tilde{S}$ is a martingale and therefore, for some predictable K, we have the martingale representation

$$\tilde{S}_t = S_0 + \int_0^t K_s d\tilde{W}_s.$$

And by Ito's lemma, using $f(\tilde{S}_t) = B_t\tilde{S}_t = S_t$,

$$S_t = S_0 + \int_0^t B_s d\tilde{S}_s = S_0 + \int_0^t B_s K_s d\tilde{W}_s.$$

Let $X = H/K$ and $Y = B^{-1}(M - XS)$ where G denotes the gains of the trading strategy (X,Y). By definition of the value process,

$$V_t = X_tS_t + Y_tB_t = X_tS_t + \frac{M_t - X_tS_t}{B_t}B_t = M_t.$$

And in particular, at time T,

$$V_T = M_T = B_T \mathbb{E}_Q\{B_T^{-1} C_T | \mathcal{F}_T\} = B_T B_T^{-1} C_T \mathbb{E}_Q\{\mathbb{1}|\mathcal{F}_T\} = C_T.$$

Therefore (X, Y) is a replicating portfolio for C. Observe that

$$V_0 + \int_0^t X_s d\tilde{S}_s = V_0 + \int_0^t X_s K_s d\tilde{W}_s = V_0 + \int_0^t H_s d\tilde{W}_s = V_0 + B_t^{-1} M_t - M_0 = \tilde{V}_t.$$

And by proposition 1 we see that (X, Y) is self-financing. We may conclude that $M_t = C_t$, or else there is an arbitrage opportunity. In particular, if $M_0 < C_0$ then the strategy $(X, Y, -M_0/C_0)$ in the extended market consisting of S, B, and C has $V_0 = M_0 - (M_0/C_0)C_0 = 0$, but

$$V_T = M_T - (M_0/C_0)C_T = C_T - (M_0/C_0)C_T = C_T(1 - M_0/C_0) > 0. \qquad \square$$

3. Example

Suppose C is a european call with strike K and expiry T on an underlying asset S. We assume that S walks like geometric brownian motion under the physical measure P. That is, for constants μ and σ, and $W_t \sim \mathcal{N}(0, t)$,

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

The payoff of the call is given by $C_T = \max(S_T - K, 0)$. By proposition 2.2, the present value of C is given by the discounted expectation of its payoff under an equivalent martingale measure Q:

$$C_0 = e^{-rT} \mathbb{E}_Q\{\max(S_T - K, 0)\}.$$

Writing this expression in terms of the max function makes it difficult to work with; with some simple algebra we can transform the equation into a more useful form:

Proposition 3.1. *Under an equivalent martingale measure Q, the price of a european call is given by*

$$C_0 = e^{-rT} \mathbb{E}_Q\{S_T \mathbb{1}_{\{S_T > K\}}\} - e^{-rT} K Q(S_T > K).$$

Proof.

$$C_0 = e^{-rT} \mathbb{E}_Q\{\max(S_T - K, 0)\} = e^{-rT} \mathbb{E}_Q\{(S_T - K)\mathbb{1}_{\{S_T > K\}}\}$$
$$= e^{-rT} \mathbb{E}_Q\{S_T \mathbb{1}_{\{S_T > K\}} - K \mathbb{1}_{\{S_T > K\}}\} = e^{-rT} \mathbb{E}_Q\{S_T \mathbb{1}_{\{S_T > K\}}\} - e^{-rT} K \mathbb{E}_Q\{\mathbb{1}_{\{S_T > K\}}\}$$
$$= e^{-rT} \mathbb{E}_Q\{S_T \mathbb{1}_{\{S_T > K\}}\} - e^{-rT} K Q(S_T > K).$$

$\square$

It is clear from this expression that we have two difficult terms to evaluate: $\mathbb{E}_Q\{S_T \mathbb{1}_{\{S_T > K\}}\}$ and $Q(S_T > K)$. Our work will proceed as follows. First, we will describe the evolution of the asset S (proposition 2). Next we will establish the existence of an equivalent martingale measure Q and characterize the evolution of S under Q (proposition 3). We will then use our results about S to characterize the space $\{S_T > K\}$ (proposition 4). Finally, we will tackle the difficult terms individually (propositions 5 and 6). Putting these results together will yield the Black-Scholes formula (theorem 7).

Proposition 3.2. *Let $W_t \sim \mathcal{N}(0,t)$. If a process S walks like geometric brownian motion then*

$$S_t = S_0 \exp\big((\mu - \sigma^2/2)t + \sigma W_t\big).$$

Proof. First, observe that the given differential dynamics of S_t are properly understood as a notational shorthand for the integral

$$S_t = S_0 + \mu \int_0^t S_t dt + \sigma \int_0^t S_t dW_t.$$

The former integral is newtonian and easily handled. The latter is stochastic and requires more care. By Ito's lemma,

$$f(S_t) - f(S_0) = \int_0^t f'(S_t) dS_t + \frac{1}{2} \int_0^t f''(S_t) d[S,S]_t.$$

Substituting $f(x) = \log(x)$ we have

$$\log(S_t/S_0) = \int_0^t \frac{dS_t}{S_t} - \frac{1}{2} \int_0^t \frac{d[S,S]_t}{S_t^2}.$$

And by the integral dynamics of S_t, linearity of the stochastic integral, and associativity of the stochastic integral respectively

$$\int_0^t \frac{dS_t}{S_t} = \int_0^t \frac{1}{S_t} d\left(\mu \int_0^t S_t dt + \sigma \int_0^t S_t dW_t\right)$$

$$= \int_0^t \frac{1}{S_t} d\left(\mu \int_0^t S_t dt\right) + \int_0^t \frac{1}{S_t} d\left(\sigma \int_0^t S_t dW_t\right) = \mu \int_0^t dt + \sigma \int_0^t dW_t = \mu t + \sigma W_t.$$

As an aside, observe how formally substituting the (formal) differential dynamics for S_t yields the same result. We can also view associativity as a formal cancellation of differential and integral operators. By properties of quadratic variation, the above calculation, and Levy characterization

$$\int_0^t \frac{d[S,S]_t}{S_t^2} = \left[\int_0^t \frac{dS_t}{S_t}, \int_0^t \frac{dS_t}{S_t}\right]$$

$$= [\mu t, \mu t] + 2[\mu t, \sigma W_t] + [\sigma W_t, \sigma W_t] = \sigma^2 [W_t, W_t] = \sigma^2 t.$$

We therefore have

$$\log(S_t/S_0) = \mu t + \sigma W_t - \frac{\sigma^2}{2} t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

And from this the result trivially follows. $\square$

Proposition 3.3. *Suppose S walks like geometric Brownian motion under P. Then S admits an equivalent martingale measure Q. Specifically, let $\tilde{W}_t = W_t - \frac{r-\mu}{\sigma} t$. Then $\tilde{W}_t$ is Q Brownian motion and $dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$.*

Proof. Let Q be equivalent to P and define $Z = dQ/dP$, $Z_t = \mathbb{E}\{Z|\mathcal{F}_t\}$. Clearly Z_t is a P martingale and by martingale representation, we may write Z_t as an integral against Brownian motion for some predictable process J_t:

$$Z_t = 1 + \int_0^t J_s dW_s.$$

Assuming Z is well-behaved, we can find H_s such that $Z_s H_s = J_s$. Therefore

$$Z_t = 1 + \int_0^t Z_s H_s dW_s.$$

Letting $N = H \cdot W$, by definition of the stochastic exponential, $Z_t = \mathcal{E}(N)_t$. By Girsanov's Theorem, the following expression is a Q local martingale:

$$\int_0^t \sigma S_s dW_s - \int_0^t \frac{1}{Z_s} d\left[Z, \int_0^\cdot \sigma S_\tau dW_\tau\right]_s.$$

Our earlier work shows that

$$\left[Z, \int_0^\cdot \sigma S_\tau dW_\tau\right]_s = \left[\int_0^\cdot Z_\tau H_\tau dW_\tau, \int_0^\cdot \sigma S_\tau dW_\tau\right]_s$$

And by properties of quadratic variation and Levy's theorem,

$$\int_0^t \frac{1}{Z_s} d\left[Z, \int_0^\cdot \sigma S_\tau dW_\tau\right]_s = \int_0^t \frac{1}{Z_s} Z_s H_s \sigma S_s d[W,W]_s = \int_0^t H_s \sigma S_s ds.$$

Setting $H_t = \frac{r-\mu}{\sigma}$, we see that the following is a Q local martingale:

$$\int_0^t \sigma S_s dW_s - \int_0^t H_s \sigma S_s ds = \int_0^t \mu S_s dt + \int_0^t \sigma S_s dW_s - \int_0^t r S_s ds = S_t - \int_0^t r S_s ds.$$

We refer to H as the market price of risk, or the Sharpe ratio. Observe that

$$\int_0^t \frac{1}{Z_s} d[Z,W]_s = \int_0^t \frac{1}{Z_s} d\left[\int_0^\cdot Z_\tau H_\tau dW_\tau, \int_0^\cdot dW_\tau\right]_s = \int_0^t H_s d[W,W]_s.$$

And by Girsanov's theorem we proceed to define another Q local martingale

$$\tilde{W}_t = W_t + \int_0^t \frac{1}{Z_s} d[Z,W]_s = W_t + \int_0^t H_t dt = W_t + \frac{r-\mu}{\sigma} t.$$

Clearly $\tilde{W}_t$ is continuous, $[\tilde{W}_t, \tilde{W}_t] = [W_t, W_t] = t$, and so by Levy characterization $\tilde{W}_t$ is Brownian motion under Q. Furthermore, by linearity

$$\int_0^t \sigma S_t d\tilde{W}_t + \int_0^t r S_s ds = \int_0^t \sigma S_t dW_t + \int_0^t \mu S_t ds = S_t.$$

This of course can be expressed differentially as

$$dS_t = r S_t dt + \sigma S_t d\tilde{W}_t.$$

Brownian motion is square-integrable and therefore Q is a spot martingale measure. The result follows by proposition 2.2. $\square$

Proposition 3.4. *Let* $d_0 = \frac{1}{\sigma\sqrt{T}}\left(\log(K/S_0) - (\mu - \sigma^2/2)T\right)$. *Then*

$$\{S_T > K\} = \left\{\frac{W_T}{\sqrt{T}} > d_0\right\}.$$

Proof. By Proposition 2,

$$\{S_T > K\} = \{S_0 \exp\left((\mu - \sigma^2/2)T + \sigma W_T\right) > K\}.$$

And solving for the random variable W_T gives us

$$\{S_T > K\} = \left\{W_T > \frac{1}{\sigma}\left(\log(K/S_0) - (\mu - \sigma^2/2)T\right)\right\}.$$

The result follows, dividing by $\sqrt{T}$ (it is more convenient to work with a unit-variance random variable). $\square$

Proposition 3.5. *Let* $d_2 = \frac{1}{\sigma\sqrt{T}}\left(\log(S_0/K) + (r - \sigma^2/2)T\right)$. *Then*

$$Q(S_T > K) = \mathcal{N}(d_2; 0, 1).$$

Proof. By propositions 4 and 3 respectively we have

$$Q(S_T > K) = Q\left(\frac{W_T}{\sqrt{T}} > d_0\right) = Q\left(\frac{\tilde{W}_T + \left(\frac{r-\mu}{\sigma}\right)T}{\sqrt{T}} > d_0\right)$$

$$= Q\left(\frac{\tilde{W}_T}{\sqrt{T}} > d_0 + \frac{(r-\mu)T}{\sigma\sqrt{T}}\right) = Q\left(\frac{\tilde{W}_T}{\sqrt{T}} > d_2\right) = \mathcal{N}(d_2; 0, 1).$$

$\square$

Proposition 3.6. *Let* $d_1 = d_2 + \sigma\sqrt{T}$, *where* d_2 *is defined as in Proposition* 5. *Then*

$$\mathbb{E}_Q\{S_T \mathbb{1}_{\{S_T > K\}}\} = S_0 e^{rT}\mathcal{N}(d_2 + \sigma\sqrt{T}; 0, 1).$$

Proof. Expanding the expectation,

$$\mathbb{E}_Q\{S_T \mathbb{1}_{\{S_T > K\}}\} = \int_{S_T > K} S_T dQ = \int_\Omega \mathbb{1}_{\{S_T > K\}} S_T dQ.$$

By proposition 4 we can substitute above with

$$\int_\Omega \mathbb{1}_{\{S_T > K\}} S_T dQ = \int_\Omega \mathbb{1}_{\left\{W_T > \frac{1}{\sigma}(\log(K/S_0) - (\mu - \sigma^2/2)T)\right\}} S_0 \exp\left((\mu - \sigma^2/2)T + \sigma W_T\right) dQ$$

Let $U_T = \tilde{W}_T/\sqrt{T}$. Then changing variables we have

$$\int_\Omega \mathbb{1}_{\{S_T > K\}} S_T dQ = \int_\Omega \mathbb{1}_{\{U_T > d2\}} S_0 \exp\left((r - \sigma^2/2)T + \sigma\sqrt{T}U_T\right) dQ.$$

By the law of the unconscious statistician ($d(U_*Q)/dx$ denotes the density of U under Q)

$$\int_\Omega \mathbb{1}_{\{S_T > K\}} S_T dQ = \int_{\mathbb{R}} \mathbb{1}_{\{x > d2\}} S_0 \exp\left((r - \sigma^2/2)T + \sigma\sqrt{T}x\right) \frac{d(U_*Q)}{dx} dx.$$

And because $U_T \sim \mathcal{N}(0,1)$ under Q we have

$$\begin{aligned}\int_\Omega \mathbb{1}_{\{S_T>K\}} S_T dQ &= \int_{-d_2}^{\infty} S_0 \exp\left((r-\sigma^2/2)T + \sigma\sqrt{T}x\right)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &= S_0 e^{rT}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(\sigma\sqrt{T}x - \frac{x^2}{2} - \frac{\sigma^2}{2}T\right)dx.\end{aligned}$$

Let $u = x - \sigma\sqrt{T}$. Then

$$\begin{aligned}\mathbb{E}_Q\{S_T\mathbb{1}_{\{S_T>K\}}\} &= S_0 e^{rT}\int_{-d_2-\sigma\sqrt{T}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \\ &= S_0 e^{rT}\left(1 - \mathcal{N}(-d_2 - \sigma\sqrt{T}; 0, 1)\right) \\ &= S_0 e^{rT}\mathcal{N}(d_2 + \sigma\sqrt{T}; 0, 1)\end{aligned}$$

□

Theorem 3.7. *(Black-Scholes) Let d_2 and d_1 be defined as in Propositions 4 and 6 respectively. Then*

$$P = S_0\mathcal{N}(d_1; 0, 1) - e^{-rT}K\mathcal{N}(d_2; 0, 1).$$

Proof. This result follows trivially from Propositions 1, 5, and 6:

$$\begin{aligned}P &= e^{-rT}\mathbb{E}_Q\{S_T\mathbb{1}_{\{S_T>K\}}\} - e^{-rT}KQ(S_T > K) \\ &= e^{-rT}S_0 e^{rT}\mathcal{N}(d_2 + \sigma\sqrt{T}; 0, 1) - e^{-rT}K\mathcal{N}(d_2; 0, 1) \\ &= S_0\mathcal{N}(d_2 + \sigma\sqrt{T}; 0, 1) - e^{-rT}K\mathcal{N}(d_2; 0, 1).\end{aligned}$$

□

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