BLACK SCHOLES VIA PDES

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1. introduction

This is the third in a series of three papers examining the Black-Scholes option model from different perspectives. This one takes the perspective of PDEs. It is the perspective taken in the inital derivation of the formula in [**?**]. That paper, however, glosses over nearly all of the mathematical details of the derivation. We take a more leisurely approach here.

2. the setup

Let S be a continuously traded stock and B be a risk-free bond. If I hold X amount of S and Y amount of B over a period from 0 to t then the **gains** of my portfolio over $[0,t]$ are given by the stochastic integral

$$G_t = \int_0^t X_s dS_s + \int_0^t Y_s dB_s.$$

We call (X,Y) a **trading strategy** and require that X,Y be predictable processes. The **value** of a trading strategy at time t is just a linear combination

$$V_t = X_t S_t + Y_t B_t.$$

And we say that a strategy is **self-financing** iff $V_t = V_0 + G_t$ for all t. We can think of self-financing as a conservation principle. Combining the value process with the gain process, we see that a strategy is self-financing iff

$$dV_t = dG_t = X_t dS_t + Y_t dB_t.$$

Let $C(t,S_t)$ the price at time t of a terminal claim on S. By Ito's lemma,

$$C(t,S_t) = C(0,S_0) + \int_0^t \frac{\partial C}{\partial s} ds + \int_0^t \frac{\partial C}{\partial S} dS_s + \frac{1}{2}\int_0^t \frac{\partial^2 C}{\partial S^2} d[S,S]_s.$$

Assuming S is governed by geometric brownian motion then

$$\int_0^t \frac{\partial C}{\partial S} dS_s = \mu \int_0^t S_s \frac{\partial C}{\partial S} ds + \sigma \int_0^t S_s \frac{\partial C}{\partial S} dW_s.$$

And the second order term reduces to

$$\int_0^t \frac{\partial^2 C}{\partial S^2} d\left[\int_0^\cdot \sigma S_\tau dW_\tau, \int_0^\cdot \sigma S_\tau dW_\tau\right]_s$$

$$= \int_0^t \frac{\partial^2 C}{\partial S^2} d \int_0^s \sigma^2 S_\tau^2 d[W,W]_\tau = \sigma^2 \int_0^t S_s^2 \frac{\partial^2 C}{\partial S^2} ds.$$

Putting this together gives us

$$C(t,S_t) = C(0,S_0) + \int_0^t \frac{\partial C}{\partial s} ds + \mu \int_0^t S_s \frac{\partial C}{\partial S} ds + \sigma \int_0^t S_s \frac{\partial C}{\partial S} dW_s + \frac{1}{2}\sigma^2 \int_0^t S_s^2 \frac{\partial^2 C}{\partial S^2} ds$$

Or alternatively, in differential notation,

$$dC_t = \left(\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S_t \frac{\partial C}{\partial S} dW_t.$$

We say that (X,Y) is a **replicating strategy** (for C) iff $C(S_t) = V_t$. We define an **arbitrage opportunity** to be a self-financing trading strategy for which $V_0 = 0$, $V_t \geq 0$, and $\mathbb{E}(V_t) > 0$ for some $t > 0$. If (X,Y) replicates C then we must have $C_t = V_t$ or else the value of the strategy (X,Y,C) at time t would be $V_t - C_t > 0$, an arbitrage opportunity. Therefore, in the absence of arbitrage, assuming a replicating strategy exists, $C_0 = C(0,S_0) = V_0$. Assuming that B's evolution is governed by $dB_t = rB_t dt$ then

$$dV_t = X_t dS_t + Y_t dB_t = (\mu X_t S_t + rY_t B_t)dt + \sigma X_t S_t dW_t.$$

To replicate V, we set $dC = dV$. Equating brownian coefficients gives us $X_t = \frac{\partial C}{\partial S}$. And equating newtonian coefficients, substituting this value, we have

$$\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} = \mu S_t \frac{\partial C}{\partial S} + rY_t B_t.$$

Rearranging the portfolio value equation,

$$B_t = \frac{V_t - X_t S_t}{Y_t} = \frac{C_t - \frac{\partial C}{\partial S} S_t}{Y_t}.$$

And it follows that

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S^2} = r\left(C_t - \frac{\partial C}{\partial S} S_t \right).$$

Observe that this is just the celebrated Black-Scholes pde:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S^2} + rS_t \frac{\partial C}{\partial S} - rC_t = 0.$$

3. The Parabolic Form

We make the following substitutions, from which we will see that the Black-Scholes equation is a parabolic equation:

$$S = Ke^x, \;\; C(t,S) = Kv(x,\tau), \;\; \tau = (T-t)\sigma^2/2.$$

The partials of C under this substitution are

$$\frac{\partial C}{\partial t} = K\frac{\partial v}{\partial t} = K\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{K\sigma^2}{2}\frac{\partial v}{\partial \tau},$$

$$\frac{\partial C}{\partial S} = K\frac{\partial v}{\partial S} = K\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{K}{S}\frac{\partial v}{\partial x},$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{K}{S}\frac{\partial v}{\partial x}\right) = \frac{K}{S}\frac{\partial}{\partial S}\frac{\partial v}{\partial x} - \frac{K}{S^2}\frac{\partial v}{\partial x}$$
$$= -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S}\frac{\partial^2 v}{\partial x^2}\frac{\partial x}{\partial S} = -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2}.$$

And therefore the Black-Scholes equation reduces to

$$-\frac{K\sigma^2}{2}\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 S^2\left(-\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2}\right) + rS\frac{K}{S}\frac{\partial v}{\partial x} - rKv = 0.$$

Rearranging terms,

$$\frac{\partial v}{\partial \tau} = -\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + \frac{2r}{\sigma^2}\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{dv}{dx} - \frac{2r}{\sigma^2}v.$$

And setting $k = 2r/\sigma^2$ leaves us with

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{dv}{dx} - kv.$$

This is a parabolic partial differential equation. Parabolic equations can always be reduced to a diffusion equation; we will perform this reduction in the next section.

4. THE DIFFUSION EQUATION

We make the following substitution:

$$v(x,\tau) = e^{\alpha x + \beta\tau}h(x,\tau).$$

Recomputing our partials under this substitution gives us

$$\frac{\partial v}{\partial x} = e^{\alpha x+\beta\tau}\left(\alpha h(x,\tau) + \frac{\partial h}{\partial x}\right),$$
$$\frac{\partial v}{\partial \tau} = e^{\alpha x+\beta\tau}\left(\beta h(x,\tau) + \frac{\partial h}{\partial \tau}\right),$$
$$\frac{\partial^2 v}{\partial x^2} = e^{\alpha x+\beta\tau}\left(\alpha^2 h(x,\tau) + 2\alpha\frac{\partial h}{\partial x} + \frac{\partial^2 h}{\partial x^2}\right).$$

Our Black-Scholes equation therefore becomes

$$\beta h(x,\tau) + \frac{\partial h}{\partial \tau} = a^2 h(x,\tau) + 2\alpha\frac{\partial h}{\partial x} + \frac{\partial^2 h}{\partial x^2} + (k-1)\left(\alpha h(x,\tau) + \frac{\partial h}{\partial x}\right) - kh(x,\tau).$$

And simple algebra leaves us with

$$\frac{\partial h}{\partial \tau} = (\alpha^2 + (k-1)\alpha - k - \beta)h(x,\tau) + (2\alpha + k - 1)\frac{\partial h}{\partial x} + \frac{\partial^2 h}{\partial x^2}.$$

Choosing α, β such that the coefficients above vanish, we require that

$$\alpha = \frac{1-k}{2}.$$

And furthermore that

$$\beta = \alpha^2 + (k-1)\alpha - k = \frac{1-2k+k^2}{4} + \frac{-k^2+2k-1}{2} - k$$
$$= \frac{-k^2-2k-1}{4} = -\frac{(k+1)^2}{4}.$$

Our parabolic equation then reduces to a simple diffusion equation:

$$\frac{\partial h}{\partial \tau} = \frac{\partial^2 h}{\partial x^2}.$$

The diffusion equation can be solved with fourier transforms.

5. Fourier transforms

Let $\mathscr{F}$ denote the fourier transform operator and recall that

$$\mathscr{F}\frac{\partial^n f}{\partial x^n}(\omega) = (2\pi i\omega)^n \mathscr{F}f(\omega).$$

Transforming the diffusion equation with respect to x therefore gives us

$$\mathscr{F}\frac{\partial h}{\partial \tau}(\omega,\tau) = (2\pi i\omega)^2 \mathscr{F}h(\omega,\tau).$$

And by Leibniz's rule (differentiation under the integral sign) we may commute $\mathscr{F}$ and ∂ on the left-hand side:

$$\frac{\partial}{\partial \tau}\mathscr{F}h(\omega,\tau) = \mathscr{F}\frac{\partial h}{\partial \tau}(\omega,\tau) = (2\pi i\omega)^2 \mathscr{F}h(\omega,\tau).$$

These dynamics are simple exponential growth in τ and we may write

$$\mathscr{F}h(\omega,\tau) = e^{(2\pi i\omega)^2\tau}\mathscr{F}h(\omega,0).$$

It follows by the Fourier inversion theorem that

$$h(x,\tau) = \mathscr{F}^{-1}\mathscr{F}h(x,\tau) = \mathscr{F}^{-1}\left(e^{(2\pi i\omega)^2\tau}\mathscr{F}h(\omega,0)\right).$$

By the convolution theorem,

$$h(x,\tau) = h(x,0) * \mathscr{F}^{-1}e^{(2\pi i\omega)^2\tau}.$$

The latter term can be computed by completing the square:

$$\mathscr{F}^{-1}e^{-\omega^2\tau} = \int_{-\infty}^{\infty} e^{(2\pi i\omega)^2\tau}e^{2\pi i\omega x}d\omega = \int_{-\infty}^{\infty}\exp\left((2\pi i\omega)^2\tau + 2\pi i\omega x\right)d\omega$$
$$= \int_{-\infty}^{\infty}\exp\left(-\tau\left(2\pi\omega - i\frac{x}{2\tau}\right)^2 - \frac{x^2}{4\tau}\right)d\omega = e^{-x^2/4\tau}\int_{-\infty}^{\infty}\exp\left(-\tau\left(2\pi\omega - i\frac{x}{2\tau}\right)^2\right)d\omega$$

And setting $\xi = \sqrt{\tau}(2\pi\omega - ix/2\tau)$ we see that

$$\mathscr{F}^{-1}e^{-\omega^2\tau} = \frac{1}{2\pi\sqrt{\tau}}e^{-x^2/4\tau}\int_{-\infty}^{\infty}e^{-\xi^2}d\xi = \frac{1}{\sqrt{4\pi\tau}}e^{-x^2/4\tau}.$$

By the definition of convolution, we therefore have

$$h(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} h(x,0) \exp\left(-\frac{(x-\xi)^2}{4\tau}\right) d\xi.$$

6. EUROPEAN CALLS

If we take C to be a european call with expiry T and strike K, then we may impose boundary conditions on the Black-Scholes pde:

$$C(T,0) = 0, \ \lim_{S_t \to \infty} C(t,S_t) = S_t, \ C(T,S_T) = \max(S_T - K, 0).$$

Working back through our chain of substitutions,

$$h(x,0) = e^{-\alpha x} v(x,0) = \frac{1}{K} e^{-\alpha x} C(T, Ke^x) = \max(e^{(1-\alpha)x} - e^{-\alpha x}, 0).$$

Substituting this expression in our general formula for h, and observing that its support is

$$h(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} \max(e^{(1-\alpha)\xi} - e^{-\alpha\xi}, 0) \exp\left(-\frac{(x-\xi)^2}{4\tau}\right) d\xi$$

Observe that $e^{(1-\alpha)\xi} - e^{-\alpha\xi} > 0$ if and only if $\xi > 0$ and therefore

$$h(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{0}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4\tau} + (1-\alpha)\xi\right) d\xi - \frac{1}{\sqrt{4\pi\tau}} \int_{0}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4\tau} - \alpha\xi\right) d\xi.$$

Tackling the second integral, completing the square we have

$$\int_{0}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4\tau} - \alpha\xi\right) d\xi = e^{-\alpha(x-\alpha\tau)} \int_{0}^{\infty} \exp\left(\frac{-(x-2\tau\alpha-\xi)^2}{4\tau}\right) d\xi$$

Changing variables to $\zeta = (x - 2\tau\alpha - \xi)/\sqrt{2\tau}$ then $d\zeta = -d\xi/\sqrt{2\tau}$ and the integral above becomes

$$-\sqrt{2\tau} \int_{\zeta(0)}^{\zeta(\infty)} e^{-\zeta^2/2} d\zeta = \sqrt{2\tau} \int_{-\infty}^{\zeta(0)} e^{-\zeta^2/2} d\zeta.$$

And combining our work we have

$$-\frac{1}{\sqrt{4\pi\tau}} \int_{0}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4\tau} - \alpha\xi\right) d\xi = e^{-\alpha(x-\alpha\tau)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta(0)} e^{-\zeta^2/2} d\zeta$$

$$= e^{-\alpha(x-\alpha\tau)} \mathcal{N}\left(\frac{x-2\tau\alpha}{\sqrt{2\tau}}; 0, 1\right).$$

Similarly, we get

$$\frac{1}{\sqrt{4\pi\tau}} \int_{0}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4\tau} + (1-\alpha)\xi\right) d\xi = e^{(1-\alpha)(x+(1-\alpha)\tau)} \mathcal{N}\left(\frac{x+2\tau(1-\alpha)}{\sqrt{2\tau}}; 0, 1\right).$$

Further undwinding our substitutions, observe that

$$C(t,S) = Kv(x,\tau) = Ke^{\alpha x + \beta\tau} h(x,\tau).$$

Recalling our definitions of α and β we see that

$$\alpha x + \beta\tau - \alpha(x - \alpha\tau) = (\beta + \alpha^2)\tau = \frac{(1-k)^2 - (k+1)^2}{4}\tau = -k\tau.$$

And we also have

$$\alpha x + \beta\tau + (1-\alpha)(x + (1-\alpha)\tau) = \beta\tau + x + (1-\alpha)^2\tau$$
$$= x - \frac{(k+1)^2}{4}\tau + \left(1 - \frac{1-k}{2}\right)^2\tau = x - \frac{(k+1)^2}{4}\tau + \frac{(1+k)^2}{4}\tau = x.$$

And making use of these computations,

$$C(t,S) = Ke^x\mathcal{N}\left(\frac{x - 2\tau\alpha}{\sqrt{2\tau}}; 0, 1\right) - Ke^{-k\tau}\mathcal{N}\left(\frac{x + 2\tau(1-\alpha)}{\sqrt{2\tau}}; 0, 1\right).$$

Expanding the normal terms, we see that

$$x + 2\tau(1-\alpha) = \log(S/K) + (T-t)\sigma^2\left(1 - \frac{1 - 2r/\sigma^2}{2}\right)$$
$$= \log(S/K) + (T-t)(r + \sigma^2/2).$$

We define d_2 by the expression

$$d_2 = \frac{x + 2\tau(1-\alpha)}{\sqrt{2\tau}} = \frac{\log(S/K) + (T-t)(r + \sigma^2/2)}{\sigma\sqrt{T-t}}.$$

Similarly we have

$$x - 2\tau\alpha = \log(S/K) - (T-t)\sigma^2\frac{1 - 2r/\sigma^2}{2}$$
$$= \log(S/K) + (T-t)(r - \sigma^2/2).$$

And likewise we define d_1 by

$$d_1 = \frac{x - 2\tau\alpha}{\sqrt{2\tau}} = \frac{\log(S/K) + (T-t)(r - \sigma^2/2)}{\sigma\sqrt{T-t}}.$$

From this we deduce the Black-Scholes formula:

$$C(t,S) = Ke^x\mathcal{N}(d_1; 0, 1) - Ke^{-k\tau}\mathcal{N}(d_2; 0, 1).$$
$$= S\mathcal{N}(d_1; 0, 1) - Ke^{-r(T-t)}\mathcal{N}(d_2; 0, 1).$$

References

[1] Black, Fischer, and Myron Scholes. "The pricing of options and corporate liabilities." The journal of political economy (1973): 637-654.
[2] http://www.francoiscoppex.com/blackscholes.pdf