BLACK SCHOLES VIA PDES

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1. INTRODUCTION

This is the third in a series of three papers examining the Black-Scholes option model from different perspectives. This one takes the perspective of PDEs. It is the perspective taken in the initial derivation of the formula in \([?]\). That paper, however, glosses over nearly all of the mathematical details of the derivation. We take a more leisurely approach here.

2. THE SETUP

Let \( S \) be a continuously traded stock and \( B \) be a risk-free bond. If I hold \( X \) amount of \( S \) and \( Y \) amount of \( B \) over a period from 0 to \( t \) then the gains of my portfolio over \([0, t]\) are given by the stochastic integral

\[
G_t = \int_0^t X_s dS_s + \int_0^t Y_s dB_s.
\]

We call \((X, Y)\) a trading strategy and require that \(X, Y\) be predictable processes. The value of a trading strategy at time \( t \) is just a linear combination

\[
V_t = X_t S_t + Y_t B_t.
\]

And we say that a strategy is self-financing iff \( V_t = V_0 + G_t \) for all \( t \). We can think of self-financing as a conservation principle. Combining the value process with the gain process, we see that a strategy is self-financing iff

\[
dV_t = dG_t = X_t dS_t + Y_t dB_t.
\]

Let \( C(t, S_t) \) the price at time \( t \) of a terminal claim on \( S \). By Ito’s lemma,

\[
C(t, S_t) = C(0, S_0) + \int_0^t \frac{\partial C}{\partial s} ds + \int_0^t \frac{\partial C}{\partial S} dS_s + \frac{1}{2} \int_0^t \frac{\partial^2 C}{\partial S^2} d[S, S]_s.
\]

Assuming \( S \) is governed by geometric brownian motion then

\[
\int_0^t \frac{\partial C}{\partial S} dS_s = \mu \int_0^t S_s \frac{\partial C}{\partial S} ds + \sigma \int_0^t S_s \frac{\partial C}{\partial S} dW_s.
\]

And the second order term reduces to

\[
\int_0^t \frac{\partial^2 C}{\partial S^2} d\left[ \int_0^t \sigma_s dW_s, \int_0^t \sigma_s dW_s \right]_s.
\]
\[
\int_0^t \frac{\partial^2 C}{\partial S^2} d \int_0^{\tau} \sigma^2 S^2 r d[W, W] \tau = \sigma^2 \int_0^t S^2 \frac{\partial^2 C}{\partial S^2} ds.
\]

Putting this together gives us

\[
C(t, S_t) = C(0, S_0) + \int_0^t \frac{\partial C}{\partial s} ds + \mu \int_0^t S \frac{\partial C}{\partial S} ds + \sigma \int_0^t S \frac{\partial C}{\partial S} dW_s + \frac{1}{2} \sigma^2 \int_0^t S^2 \frac{\partial^2 C}{\partial S^2} ds.
\]

Or alternatively, in differential notation,

\[
dC_t = \left( \frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW_t.
\]

We say that \((X, Y)\) is a replicating strategy (for \(C\)) iff \(C(S_t) = V_t\). We define an arbitrage opportunity to be a self-financing trading strategy for which \(V_0 = 0, V_t \geq 0, \) and \(E(V_t) > 0\) for some \(t > 0\). If \((X, Y)\) replicates \(C\) then we must have \(C_t = V_t\) or else the value of the strategy \((X, Y, C)\) at time \(t\) would be \(V_t - C_t > 0\), an arbitrage opportunity. Therefore, in the absence of arbitrage, assuming a replicating strategy exists, \(C_0 = C(0, S_0) = V_0\). Assuming that \(B\)'s evolution is governed by \(dB_t = rB_t dt\) then

\[
dV_t = X_t dS_t + Y_t dB_t = (\mu X_t S_t + rY_t B_t) dt + \sigma X_t S_t dW_t.
\]

To replicate \(V\), we set \(dC = dV\). Equating brownian coefficients gives us \(X_t = \frac{\partial C}{\partial S}\). And equating newtonian coefficients, substituting this value, we have

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r \left( C_t - \frac{\partial C}{\partial S} S_t \right).
\]

Rearranging the portfolio value equation,

\[
B_t = \frac{V_t - X_t S_t}{Y_t} = C_t - \frac{\partial C}{\partial S} S_t.
\]

And it follows that

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r \left( C_t - \frac{\partial C}{\partial S} S_t \right).
\]

Observe that this is just the celebrated Black-Scholes pde:

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC_t = 0.
\]

3. THE PARABOLIC FORM

We make the following substitutions, from which we will see that the Black-Scholes equation is a parabolic equation:

\[
S = Ke^x, \ C(t, S) = K v(x, \tau), \ \tau = (T - t)\sigma^2/2.
\]

The partials of \(C\) under this substitution are

\[
\frac{\partial C}{\partial t} = K \frac{\partial v}{\partial t} = K \frac{\partial v}{\partial \tau} \frac{\partial \tau}{\partial t} = -K \frac{\sigma^2}{2} \frac{\partial v}{\partial \tau},
\]

\[
\frac{\partial C}{\partial S} = K \frac{\partial v}{\partial S} = K \frac{\partial v}{\partial x} \frac{\partial x}{\partial S} = \frac{K \partial v}{S \partial x}.
\]
\[
\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S} \left( \frac{K}{S} \frac{\partial v}{\partial x} \right) = \frac{K}{S} \frac{\partial}{\partial S} \frac{\partial v}{\partial x} - \frac{K}{S^2} \frac{\partial v}{\partial S}
\]

And therefore the Black-Scholes equation reduces to

\[
-K \sigma^2 \frac{\partial v}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \left( -\frac{K}{S^2} \frac{\partial v}{\partial x} + \frac{K}{S^2} \frac{\partial^2 v}{\partial x^2} \right) + r S K \frac{\partial v}{\partial S} - r K v = 0.
\]

Rearranging terms,

\[
\frac{\partial v}{\partial \tau} = -\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + \frac{2r}{\sigma^2} \frac{\partial v}{\partial x} - \frac{2r}{\sigma^2} v = \frac{\partial^2 v}{\partial x^2} + \left( \frac{2r}{\sigma^2} - 1 \right) \frac{dv}{dx} - \frac{2r}{\sigma^2} v.
\]

And setting \( k = 2r/\sigma^2 \) leaves us with

\[
\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1) \frac{dv}{dx} - kv.
\]

This is a parabolic partial differential equation. Parabolic equations can always be reduced to a diffusion equation; we will perform this reduction in the next section.

4. THE DIFFUSION EQUATION

We make the following substitution:

\[
v(x, \tau) = e^{\alpha x + \beta \tau} h(x, \tau).
\]

Recomputing our partials under this substitution gives us

\[
\frac{\partial v}{\partial x} = e^{\alpha x + \beta \tau} \left( \alpha h(x, \tau) + \frac{\partial h}{\partial x} \right),
\]

\[
\frac{\partial v}{\partial \tau} = e^{\alpha x + \beta \tau} \left( \beta h(x, \tau) + \frac{\partial h}{\partial \tau} \right),
\]

\[
\frac{\partial^2 v}{\partial x^2} = e^{\alpha x + \beta \tau} \left( \alpha^2 h(x, \tau) + 2\alpha \frac{\partial h}{\partial x} + \frac{\partial^2 h}{\partial x^2} \right).
\]

Our Black-Scholes equation therefore becomes

\[
\beta h(x, \tau) + \frac{\partial h}{\partial \tau} = \alpha^2 h(x, \tau) + 2\alpha \frac{\partial h}{\partial x} + \frac{\partial^2 h}{\partial x^2} + (k-1) \left( \alpha h(x, \tau) + \frac{\partial h}{\partial x} \right) - kh(x, \tau).
\]

And simple algebra leaves us with

\[
\frac{\partial h}{\partial \tau} = (\alpha^2 + (k-1)\alpha - k - \beta) h(x, \tau) + (2\alpha + k - 1) \frac{\partial h}{\partial x} + \frac{\partial^2 h}{\partial x^2}.
\]

Choosing \( \alpha, \beta \) such that the coefficients above vanish, we require that

\[
\alpha = \frac{1 - k}{2}.
\]
And furthermore that

\[
\beta = \alpha^2 + (k - 1)\alpha - k = \frac{1 - 2k + k^2}{4} + \frac{-k^2 + 2k - 1}{2} - k \\
= \frac{-k^2 - 2k - 1}{4} = -\frac{(k + 1)^2}{4}.
\]

Our parabolic equation then reduces to a simple diffusion equation:

\[
\frac{\partial h}{\partial \tau} = \frac{\partial^2 h}{\partial x^2}.
\]

The diffusion equation can be solved with Fourier transforms.

5. Fourier Transforms

Let \( \mathcal{F} \) denote the Fourier transform operator and recall that

\[
\mathcal{F} \frac{\partial^n f}{\partial x^n} (\omega) = (2\pi i \omega)^n \mathcal{F} f (\omega).
\]

Transforming the diffusion equation with respect to \( x \) therefore gives us

\[
\mathcal{F} \frac{\partial h}{\partial \tau} (\omega, \tau) = (2\pi i \omega)^2 \mathcal{F} h (\omega, \tau).
\]

And by Leibniz’s rule (differentiation under the integral sign) we may commute \( \mathcal{F} \) and \( \partial \) on the left-hand side:

\[
\frac{\partial}{\partial \tau} \mathcal{F} h (\omega, \tau) = \mathcal{F} \frac{\partial h}{\partial \tau} (\omega, \tau) = (2\pi i \omega)^2 \mathcal{F} h (\omega, \tau).
\]

These dynamics are simple exponential growth in \( \tau \) and we may write

\[
\mathcal{F} h (\omega, \tau) = e^{(2\pi i \omega)^2 \tau} \mathcal{F} h (\omega, 0).
\]

It follows by the Fourier inversion theorem that

\[
h(x, \tau) = \mathcal{F}^{-1} e^{(2\pi i \omega)^2 \tau} \mathcal{F} h (\omega, 0) = \mathcal{F}^{-1} \left( e^{(2\pi i \omega)^2 \tau} \mathcal{F} h (\omega, 0) \right).
\]

By the convolution theorem,

\[
h(x, \tau) = h(x, 0) \ast \mathcal{F}^{-1} e^{(2\pi i \omega)^2 \tau}.
\]

The latter term can be computed by completing the square:

\[
\mathcal{F}^{-1} e^{-\omega^2 \tau} = \int_{-\infty}^{\infty} e^{(2\pi i \omega)^2 \tau} e^{2\pi i \omega x} d\omega = \int_{-\infty}^{\infty} \exp \left( (2\pi i \omega)^2 \tau + 2\pi i \omega x \right) d\omega \\
= \int_{-\infty}^{\infty} \exp \left( -\tau \left( 2\pi \omega - \frac{x}{2\tau} \right)^2 - \frac{x^2}{4\tau} \right) d\omega = e^{-x^2/4\tau} \int_{-\infty}^{\infty} \exp \left( -\tau \left( 2\pi \omega - \frac{x}{2\tau} \right)^2 \right) d\omega
\]

And setting \( \xi = \sqrt{\tau} (2\pi \omega - ix/2\tau) \) we see that

\[
\mathcal{F}^{-1} e^{-\omega^2 \tau} = \frac{1}{2\pi \sqrt{\tau}} e^{-x^2/4\tau} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \frac{1}{\sqrt{4\pi \tau}} e^{-x^2/4\tau}.
\]
By the definition of convolution, we therefore have
\[ h(x, \tau) = \frac{1}{\sqrt{4\pi \tau}} \int_{-\infty}^{\infty} h(x, 0) \exp \left( -\frac{(x - \xi)^2}{4\tau} \right) d\xi. \]

6. EUROPEAN CALLS

If we take \( C \) to be a european call with expiry \( T \) and strike \( K \), then we may impose boundary conditions on the Black-Scholes pde:
\[ C(T, 0) = 0, \quad \lim_{S_t \to \infty} C(t, S_t) = S_t, \quad C(T, S_T) = \max(S_T - K, 0). \]

Working back through our chain of substitutions,
\[ h(x, 0) = e^{-\alpha x} v(x, 0) = \frac{1}{K} e^{-\alpha x} C(T, Ke^x) = \max(e^{(1-\alpha)x} - e^{-\alpha x}, 0). \]

Substituting this expression in our general formula for \( h \), and observing that its support is
\[ h(x, \tau) = \frac{1}{\sqrt{4\pi \tau}} \int_{-\infty}^{\infty} \max(e^{(1-\alpha)x} - e^{-\alpha x}, 0) \exp \left( -\frac{(x - \xi)^2}{4\tau} \right) d\xi. \]

Observe that \( e^{(1-\alpha)x} - e^{-\alpha x} > 0 \) if and only if \( \xi > 0 \) and therefore
\[ h(x, \tau) = \frac{1}{\sqrt{4\pi \tau}} \int_{0}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} + (1 - \alpha)\xi \right) d\xi - \frac{1}{\sqrt{4\pi \tau}} \int_{0}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} - \alpha \xi \right) d\xi. \]

Tackling the second integral, completing the square we have
\[ \int_{0}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} - \alpha \xi \right) d\xi = e^{-\alpha(x - \alpha \tau)} \int_{0}^{\infty} \exp \left( -\frac{(x - 2\tau \alpha - \xi)^2}{4\tau} \right) d\xi. \]

Changing variables to \( \zeta = (x - 2\tau \alpha - \xi)/\sqrt{2\tau} \) then \( d\zeta = -d\xi/\sqrt{2\tau} \) and the integral above becomes
\[ -\sqrt{2\tau} \int_{e^{-\zeta^2/2}}^{0} e^{-\zeta^2/2} d\zeta = \sqrt{2\tau} \int_{-\infty}^{e^{-\zeta^2/2}} e^{-\zeta^2/2} d\zeta. \]

And combining our work we have
\[ -\frac{1}{\sqrt{4\pi \tau}} \int_{0}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} - \alpha \xi \right) d\xi = e^{-\alpha(x - \alpha \tau)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{e^{-\zeta^2/2}} e^{-\zeta^2/2} d\zeta \]
\[ = e^{-\alpha(x - \alpha \tau)} \mathcal{N} \left( \frac{x - 2\tau \alpha}{\sqrt{2\tau}} ; 0, 1 \right). \]

Similarly, we get
\[ \frac{1}{\sqrt{4\pi \tau}} \int_{0}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} + (1 - \alpha)\xi \right) d\xi = e^{(1-\alpha)(x+\alpha \tau)} \mathcal{N} \left( \frac{x + 2\tau (1 - \alpha)}{\sqrt{2\tau}} ; 0, 1 \right). \]

Further undwinding our substitutions, observe that
\[ C(t, S) = Kv(x, \tau) = Ke^{\alpha x + \beta \tau} h(x, \tau). \]
Recalling our definitions of $\alpha$ and $\beta$ we see that

$$\alpha x + \beta \tau - \alpha(x - \alpha \tau) = (\beta + \alpha^2)\tau = \frac{(1 - k)^2 - (k + 1)^2}{4}\tau = -k\tau.$$  

And we also have

$$\alpha x + \beta \tau + (1 - \alpha)(x + (1 - \alpha)\tau) = \beta \tau + x + (1 - \alpha)^2\tau$$
$$= x - \frac{(k + 1)^2}{4}\tau + \left(1 - \frac{1 - k}{2}\right)^2 \tau = x - \frac{(k + 1)^2}{4}\tau + \frac{(1 + k)^2}{4}\tau = x.$$  

And making use of these computations,

$$C(t, S) = Ke^{x}N\left(\frac{x - 2\tau\alpha}{\sqrt{2\tau}}; 0, 1\right) - Ke^{-k\tau}N\left(\frac{x + 2\tau(1 - \alpha)}{\sqrt{2\tau}}; 0, 1\right).$$  

Expanding the normal terms, we see that

$$x + 2\tau(1 - \alpha) = \log(S/K) + (T - t)\sigma^2\left(1 - \frac{1 - 2r/\sigma^2}{2}\right)$$
$$= \log(S/K) + (T - t)(r + \sigma^2/2).$$  

We define $d_2$ by the expression

$$d_2 = \frac{x + 2\tau(1 - \alpha)}{\sqrt{2\tau}} = \frac{\log(S/K) + (T - t)(r + \sigma^2/2)}{\sigma\sqrt{T - t}}.$$  

Similarly we have

$$x - 2\tau\alpha = \log(S/K) - (T - t)\sigma^2\frac{1 - 2r/\sigma^2}{2}$$
$$= \log(S/K) + (T - t)(r - \sigma^2/2).$$  

And likewise we define $d_1$ by

$$d_1 = \frac{x - 2\tau\alpha}{\sqrt{2\tau}} = \frac{\log(S/K) + (T - t)(r - \sigma^2/2)}{\sigma\sqrt{T - t}}.$$  

From this we deduce the Black-Scholes formula:

$$C(t, S) = Ke^xN(d_1; 0, 1) - Ke^{-k\tau}N(d_2; 0, 1).$$
$$= SN(d_1; 0, 1) - Ke^{-r(T-t)}N(d_2; 0, 1).$$
References
