DENSITY NOTATION

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1. DENSITIES

Let $X$ be a random variable on $(\Omega, \mathcal{F}, P)$ with pushforward measure $X_*P$. This measure is called the law of $X$ and

$$P(X \in A) = \int_{\{X \in A\}} dP = \int_A d(X_*P).$$

If the law of $X$ is absolutely continuous with respect to Lebesgue measure, that is $X_*P \ll dx$, then there exists a Radon-Nikodym derivative $d(X_*P)/dx : \mathbb{R} \to \mathbb{R}$ such that

$$\int_A d(X_*P) = \int_A \frac{d(X_*P)}{dx} dx.$$  

This derivative is called a density and is commonly denoted by $p(x)$ and we write

$$P(X \in A) = \int_A p(x) dx.$$  

Similarly, by the law of the unconscious statistician we write

$$\mathbb{E}X = \int_{-\infty}^{\infty} xp(x) dx.$$  

Let $\mathcal{G} \subset \mathcal{F}$ be a (sub)$\sigma$-algebra. We can define a unique (up to null sets) conditional expectation $\mathbb{E}\{X|Y\}$, which for any $G \in \mathcal{G}$ satisfies

$$\int_G \mathbb{E}\{X|G\} dP = \int_G X dP.$$  

This relationship is neatly expressed by the following commutative diagram.

That is, $\mathbb{E}\{X|G\} = Y$ for some $G$-measurable $Y : \Omega \to \mathbb{R}$. Conversely, if $Y$ is a random variable with $\sigma(Y) = \mathcal{G}$ then we define $\mathbb{E}\{X|Y\} = \mathbb{E}\{X|\mathcal{G}\}$ and for some measurable $h : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}\{X|Y\} = h(Y)$. Furthermore, this allows us to define a pointwise conditional expectation $\mathbb{E}\{X|Y = y\} = h(y)$. We might want to extend this definition into a more
2. MULTIPLE VARIABLES

Now let’s consider the case of a pair of random variables \( X = (Y, Z) \). Let’s assume both \( Y \) and \( Z \) are defined on \( (\Omega, \mathcal{F}, P) \). We can define a joint density in the manner discussed above by

\[
P(Y \in A, Z \in B) = \int_{A \times B} p(y, z) d(y, z).
\]

From this definition, Fubini’s theorem, and the definition of a single-variable density,

\[
P(Y \in A) = P(X \in A \times \mathbb{R}) = \int_{A \times \mathbb{R}} p(y, z) d(y, z) = \int_A \int_{-\infty}^{\infty} p(y, z) dz dy = \int_A p(y) dy.
\]

This holds for all \( A \) and it follows that a.s.

\[
p(y) = \int_{-\infty}^{\infty} p(y, z) dz, \quad p(z) = \int_{-\infty}^{\infty} p(z, y) dz.
\]
These are called the marginals of $Y$ and $Z$. Note that we have slightly abused notation: $p(y)$ and $p(z)$ are distinct functions, distinguished lexicographically by the name of their input variable. By our definitions of densities and conditional probabilities in section 1,

$$P(Y \in A, Z \in B) = \int_B P(Y \in A | Z)d(Z_*P)$$

$$= \int_B \int_A p(y|z)dx d(Z_*P) = \int_B \int_A p(y|z)p(z)dz.$$ 

And therefore we a.s. have

$$p(y, z) = \frac{p(y|z)}{p(z)}.$$