Let $V$ be the indefinite integral operator defined by

$$Vf(t) = \int_0^t f(s)ds.$$ 

This is a linear operator. It can be defined on any domain of integrable functions, but here we restrict ourselves to domains where it behaves nicely. For example, if $f \in L^p[0,1]$ for $p \in (1, \infty]$ then by H"older’s inequality

$$|Vf(x) - Vf(y)| \leq \int_y^x |f(s)|ds \leq \|f\|_p|x-y|^{1/q}.$$ 

So $Vf$ is (H"older-)continuous on $[0,1]$. If $B_p$ is the unit ball in $L^p[0,1]$ then the image of $B_p$ in $V$ is equicontinuous. Furthermore,

$$|Vf(t)| \leq \int_0^t |f(s)|ds \leq |t|^{1/q} \leq 1.$$ 

Therefore the image of $B_p$ is (uniformly) bounded. By Arzela-Ascoli, $V : L^p[0,1] \to C[0,1]$ is compact. The preceeding argument does not go through when $V$ acts on $L^1[0,1]$. In this case equicontinuity fails, as is demonstrated by the following family $\{f_n\} \subset B_1$:

$$f_n(s) = n\mathbb{1}_{[0,1/n]}(s).$$ 

This suffices to preclude compactness of $V$; in particular, $Vf_n$ has no Cauchy subsequence.

Suppose $Vf = \lambda f$ for some $\lambda \neq 0$. By definition $\lambda f$ is (absolutely) continuous. We deduce that $Vf$ (and therefore $f$) is continuously differentiable and that $f = \lambda f'$. It follows that $f(s) = ce^{s/\lambda}$. But then $0 = \lambda f - Vf = c$ so $V$ has no eigenvalues and by the spectral theorem for compact operators, $\sigma(V) = \{0\}$.

We now turn our attention to the operator norm of $V$. For now we restrict $V$ to the square integrable domain $L^2$. Note that $C[0,1] \subset L^\infty[0,1] \subset L^2[0,1]$ and the identity mapping $I : C[0,1] \to L^2[0,1]$ is a bounded linear operator. It follows that $IV : L^2[0,1] \to L^2[0,1]$ is compact and we will proceed to consider $V : L^2[0,1] \to L^2[0,1]$. Recall that $L^2[0,1]$ is a Hilbert space with an inner product defined by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$ 

If $\|f\| \leq 1$ then by Cauchy-Schwartz

$$\|Vf\|^2 = \langle V^*Vf, f \rangle \leq \|V^*Vf\| \|f\| \leq \|V^*V\| \leq \|V^*\| \|V\|.$$
Therefore $||V|| \leq ||V^*||$ and by a symmetric argument, $||V^*|| \leq ||V||$. It follows that $||V||^2 = ||V^*V||$ and we can compute $||V||$ in terms of $V^*V$. The adjoint of $V$ is

$$V^* f(t) = \int_t^1 f(s) ds.$$  

This is easily verified by Fubini’s theorem:

$$\langle f, Vf \rangle = \int_0^1 \int_0^t f(t) f(s) ds dt = \int_0^1 \int_0^1 f(s) f(t) dt ds = \langle V^* f, f \rangle.$$  

Because compact operators form an ideal, $V^*V$ is compact. Clearly $V^*V$ is self-adjoint. By the spectral theorem for compact self-adjoint operators, $V^*V$ is diagonalizable and therefore its operator norm is just the magnitude of its largest eigenvalue.

Suppose $V^*V f = \lambda f, f \neq 0$. Then $f$ is continuous and $V f$ is continuously differentiable. It follows that $f \in C^2[0, 1]$ and

$$\lambda f''(x) = \frac{\partial^2}{\partial x^2} \int_x^1 \int_0^t f(s) ds dt = -\frac{\partial}{\partial x} \int_0^x f(s) ds = -f(x).$$  

Let $\omega^2 = 1/\lambda$. We deduce that eigenfunctions of $V^*V$ must be of the form

$$f(x) = a e^{i\omega x} + b e^{-i\omega x}.$$  

Furthermore, routine integration shows that

$$V^* V f(x) = a \int_x^1 \int_0^t e^{i\omega s} ds dt + b \int_x^1 \int_0^t e^{-i\omega s} ds dt = \frac{1}{\omega^2} f(x) + \frac{1}{i\omega} (a - b) x - \frac{1}{\omega^2} (ae^{i\omega} + be^{-i\omega}) - \frac{1}{i\omega} (a - b)$$

From the second term, we must have $a = b$ and $f$ therefore has the form $2a \cos(\omega x)$. From the third term, since $a \neq 0$, $\cos(i\omega) = 0$. It follows that $f$ is an eigenfunction if and only if $\omega = \frac{2n+1}{2} \pi$. The eigenvalues of $V^*V$ are therefore

$$\lambda_n = \frac{4}{(2n+1)^2 \pi^2}.$$  

Maximizing over $n$ we see that the largest eigenvalue of $V^*V$, and therefore its operator norm, is $\lambda_0 = \frac{2}{\pi}$. We conclude that $||V|| = \frac{2}{\pi}$.

The operator norm is crucially dependent upon the operator’s domain of definition. For example, consider instead $V : L^1[0, 1] \to L^1[0, 1]$. Then

$$||V f|| \leq \int_0^1 \int_0^t |f(s)| ds dt \leq \int_0^1 \int_0^1 |f(s)| ds dt = ||f||.$$  

Therefore $||V|| \leq 1$. Using the same example we used earlier to rule out compactness of $V$, as $n \to \infty$,

$$||V f_n|| = \int_0^1 \int_0^t n \mathbb{1}_{[0,1/n]}(s) ds dt = \int_0^{1/n} n t dt + \int_{1/n}^1 dt = 1 - \frac{1}{2n} \to 1.$$  

So in this case $||V|| = 1$. 


1. HILBERT-SCHMIDT OPERATORS

Let $L(U)$ denote the space of linear operators on a linear space $U$ over $F$. When $U$ is finite dimensional we can identify $U \otimes U^*$ with $L(U)$ by associating $u \otimes v^* \in U \otimes U^*$ with $T \in L(U)$ such that $T(w) = v^*(w)u$. Let $ev : U \times U^* \to F$ be the (bilinear) evaluation functional defined by $ev(u, v^*) = v^*(u)$. By universality of the tensor product this induces a map $tr : U \otimes U^* \to F$. We identify this map with a map $tr : L(U) \to F$, which we call the trace.

In infinite dimensions, $U \otimes U^*$ identifies only with finite rank operators. From the preceding discussion, we can define a Hilbert-Schmidt inner product of finite rank operators $S, T \in L(U)$,

$$\langle S, T \rangle_{HS} = \text{tr}(S^*T).$$

And we define the space of Hilbert-Schmidt operators to be the completion of the finite rank operators with respect to this inner product. Let $ev_u : U \otimes U^* \to U$ be the evaluation map on finite rank operators defined by $ev_u(T) = Tu$. This map is uniformly continuous and by universality of completion it induces a map on the Hilbert-Schmidt operators. Furthermore, if $T$ is Hilbert-Schmidt then

$$T(\alpha u + v) = \lim_{n \to \infty} T_n(\alpha u + v) = \alpha \lim_{n \to \infty} T_n u + \lim_{n \to \infty} T_n v = \alpha Tu + Tv.$$ 

Limits here are taken in the sense of the topology induced by the Hilbert-Schmidt inner product. We therefore identify Hilbert-Schmidt operators with a subspace of $L(U)$.

Suppose $H$ is a Hilbert space with (possibly uncountable) orthonormal basis $(e_i)_{i \in B}$ and let $u \in H$. If $A$ is a bounded linear operator on $H$ then $A$ is continuous and has an abstract fourier representation

$$Au = A \sum_{i \in B} \langle u, e_i \rangle e_i = \sum_{i \in B} \langle u, e_i \rangle Ae_i.$$

The latter sum converges in the sense of the operator topology. Let $e_i^* = \langle e_i, \cdot \rangle_{HS}$ be the dual basis associated with the Hilbert-Schmidt inner product. Then

$$Au = \sum_{i \in B} e_i^*(u) Ae_i = \left( \sum_{i \in B} e_i^* \otimes Ae_i \right)(u).$$

Here the latter sum converges in the Hilbert-Schmidt topology. It follows that

$$\text{tr}(A) = \text{tr} \sum_{i \in B} e_i^* \otimes Ae_i = \sum_{i \in B} \text{tr}(e_i^* \otimes Ae_i) = \sum_{i \in B} \langle Ae_i, e_i \rangle.$$

We can now compute the Hilbert-Schmidt norm of an operator $A$:

$$\|A\|_{HS}^2 = \langle A, A \rangle_{HS} = \text{tr}(A^*A) = \sum_{i \in B} \langle A^* Ae_i, e_i \rangle = \sum_{i \in B} \|Ae_i\|^2.$$ 

Observe that $\|A\|_{HS}^2 < \infty$ and therefore $\|Ae_i\| = 0$ for all but countably many $i \in B$. If we reorder $(e_i)$ as a countable sequence then $(Ae_i)$ as a sequence in $\ell^2$. Furthermore, by
Bessel’s inequality
\[ \sum_{i=1}^{\infty} |\langle u, e_i \rangle|^2 \leq \|u\|^2 < \infty. \]

Therefore \( \langle u, e_i \rangle \) is also a sequence in \( \ell^2 \). By Cauchy-Schwarz in \( \ell^2 \),
\[
\|Au\| \leq \sum_{i=1}^{\infty} \|\langle u, e_i \rangle A e_i \| = \sum_{i=1}^{\infty} |\langle u, e_i \rangle| \|A e_i\| \leq \left( \sum_{i=1}^{\infty} |\langle u, e_i \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} \|A e_i\|^2 \right)^{\frac{1}{2}}.
\]

This sum is finite from preceding calculations and moreover
\[ \|Au\|^2 \leq \|A\|_{HS} \|u\|. \]

It follows that \( \|A\|_{op} \leq \|A\|_{HS} \). Therefore Hilbert-Schmidt limits of finite rank operators are operator norm limits of finite rank operators, which are compact. We conclude that Hilbert-Schmidt operators are compact.

Does \( L^2(X \times X, \mu \otimes \mu) = L^2(X, \mu) \oplus L^2(X, \mu) \)?

Let \( X \) be a \( \sigma \)-finite measure space with \( k \in L^2(X \times X) \). We associate \( k \) (which we call a kernel) with an integral operator \( K \) by the map
\[
Ku(x) = \int_X k(x, y) u(y) dy.
\]

We will show that \( K \) is a Hilbert-Schmidt operator on \( L^2(X) \). By Fubini’s theorem, \( k(x, \cdot) \in L^2(X) \) almost everywhere. If \( k(x, \cdot) \in L^2(X) \) and \( u \in L^2(X) \) then by Cauchy-Schwarz on \( L^2(X) \),
\[
\int_X |k(x, y) u(y)| dy \leq \|k(x, \cdot)\| \|u\| < \infty.
\]

Therefore \( k(x, y) u(y) \in L^1(X) \) and \( K \) is defined on \( L^2(X) \) (modulo null sets). Another application of Fubini shows that
\[
\int_Y |Ku(x)|^2 dx \leq \int_X \|k(x, \cdot)\|^2 \|u\|^2 dx = \|u\|^2 \int_{X \times X} |k(x, y)|^2 dy \otimes dx = \|u\|^2 \|k\|^2 < \infty.
\]

It follows that \( Ku \in L^2(X) \) and we consider \( K \) as a linear operator \( K : L^2(X) \rightarrow L^2(X) \).

Our work further implies that \( \|K\| \leq \|k\| \).

Let \( i \in B \) index an orthonormal basis \( (e_i) \) of \( X \). By Bessel’s inequality
\[ \|K\|^2_{HS} = \text{tr}(K^* K) = \sum_{i \in B} \|K e_i\|^2 \leq \sum_{i \in B} \sum_{j \in B} |\langle K e_i, e_j \rangle|^2. \]

And by Fubini’s theorem again,
\[
\langle K e_i, e_j \rangle = \int_X K e_i(x) \bar{e}_j(x) dx = \int_{X \times X} k(x, y) e_i(y) \bar{e}_j(x) dy \otimes dx.
\]
Let $u_{ij}(x,y) = e_i(y)\overline{e_j(x)}$ and observe that $(u_{ij})$ forms an orthonormal basis for $L^2(X \times X)$. Then by another application of Bessel’s inequality,

$$\|K\|_{HS}^2 \leq \sum_{i \in B} \sum_{j \in B} |\langle Ke_i, e_j \rangle|^2 = \sum_{i,j \in B \times B} |\langle k, u_{ij} \rangle|^2 \leq \|k\|^2.$$ 

When $L^2(X)$ is separable Bessel’s inequality is replaced by Parseval’s identity and equality holds. We deduce that $K$ is a Hilbert-Schmidt operator and (for separable spaces) the map $k \mapsto K$ is (Hilbert-Schmidt)-isometric.

Remarkably, any Hilbert-Schmidt operator can be characterized by a kernel. Suppose $K$ is Hilbert-Schmidt. Then $K$ can be written as a Hilbert-Schmidt limit of finite rank operators $K_n$. For $u_i, v_i \in L^2(X)$, we can write

$$K_n w(x) = \sum_{i=1}^{n} u_i \otimes v_i(x) = \sum_{i=1}^{n} \int_X u_i(x)\overline{v_i}(y)w(y)dy = \int_X \sum_{i=1}^{n} u_i(x)\overline{v_i}(y)w(y)dy.$$

The kernel here is given by

$$k_n(x,y) = \sum_{i=1}^{n} u_i(x)\overline{v_i}(y).$$

Because $k_n \mapsto K_n$ is a Hilbert-Schmidt isometry and $(K_n)$ converges,

$$\|k_n - k_m\|_{L^2} = \|K_n - K_m\|_{HS} \to 0.$$

Therefore $(k_n)$ is Cauchy and by completeness of $L^2(X)$ we deduce that $k_n \to k$ for some $k \in L^2(X)$. This permits us to write

$$K w(x) = \int_X k(x,y)w(y)dy.$$