# THE VOLTERRA OPERATOR

JOHN THICKSTUN

Let $V$ be the indefinite integral operator defined by

$$Vf(t) = \int_0^t f(s)ds.$$

This is a linear operator. It can be defined on any domain of integrable functions, but here we restrict ourselves to domains where it behaves nicely. For example, if $f \in L^p[0,1]$ for $p \in (1,\infty]$ then by Hölder's inequality

$$|Vf(x) - Vf(y)| \leq \int_y^x |f(s)|ds \leq \|f\|_p |x-y|^{1/q}.$$

So $Vf$ is (Hölder-)continuous on $[0,1]$. If $B_p$ is the unit ball in $L^p[0,1]$ then the image of $B_p$ in $V$ is equicontinuous. Furthermore,

$$|Vf(t)| \leq \int_0^t |f(s)|ds \leq |t|^{1/q} \leq 1.$$

Therefore the image of $B_p$ is (uniformly) bounded. By Arzela-Ascoli, $V : L^p[0,1] \to C[0,1]$ is compact. The preceeding argument does not go through when $V$ acts on $L^1[0,1]$. In this case equicontinuity fails, as is demonstrated by the following family $\{f_n\} \subset B_1$:

$$f_n(s) = n\mathbb{1}_{[0,1/n]}(s).$$

This suffices to preclude compactness of $V$; in particular, $Vf_n$ has no Cauchy subsequence.

Suppose $Vf = \lambda f$ for some $\lambda \neq 0$. By definition $\lambda f$ is (absolutely) continuous. We deduce that $Vf$ (and therefore $f$) is continuously differentiable and that $f = \lambda f'$. It follows that $f(s) = ce^{s/\lambda}$. But then $0 = \lambda f - Vf = c$ so $V$ has no eigenvalues and by the spectral theorem for compact operators, $\sigma(V) = \{0\}$.

We now turn our attention to the operator norm of $V$. For now we restrict $V$ to the square integrable domain $L^2$. Note that $C[0,1] \subset L^\infty[0,1] \subset L^2[0,1]$ and the identity mapping $I : C[0,1] \to L^2[0,1]$ is a bounded linear operator. It follows that $IV : L^2[0,1] \to L^2[0,1]$ is compact and we will proceed to consider $V : L^2[0,1] \to L^2[0,1]$. Recall that $L^2[0,1]$ is a Hilbert space with an inner product defined by

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

If $\|f\| \leq 1$ then by Cauchy-Schwartz

$$\|Vf\|^2 = \langle V^*Vf, f\rangle \leq \|V^*Vf\|\|f\| \leq \|V^*V\| \leq \|V^*\|\|V\|.$$

Therefore $\|V\| \leq \|V^*\|$ and by a symmetric argument, $\|V^*\| \leq \|V\|$. It follows that $\|V\|^2 = \|V^*V\|$ and we can compute $\|V\|$ in terms of $V^*V$. The adjoint of $V$ is

$$V^*f(t) = \int_t^1 f(s)ds.$$

This is easily verified by Fubini's theorem:

$$\langle f, Vf \rangle = \int_0^1 f(t) \int_0^t f(s)dsdt = \int_0^1 f(s) \int_s^1 f(t)dtds = \langle V^*f, f \rangle.$$

Because compact operators form an ideal, $V^*V$ is compact. Clearly $V^*V$ is self-adjoint. By the spectral theorem for compact self-adjoint operators, $V^*V$ is diagonalizable and therefore its operator norm is just the magnitude of its largest eigenvalue.

Suppose $V^*Vf = \lambda f$, $f \neq 0$. Then $f$ is continuous and $Vf$ is continuously differentiable. It follows that $f \in C^2[0,1]$ and

$$\lambda f''(x) = \frac{\partial^2}{\partial x^2} \int_x^1 \int_0^t f(s)dsdt = -\frac{\partial}{\partial x} \int_0^x f(s)ds = -f(x).$$

Let $\omega^2 = 1/\lambda$. We deduce that eigenfunctions of $V^*V$ must be of the form

$$f(x) = ae^{i\omega x} + be^{-i\omega x}.$$

Furthermore, routine integration shows that

$$V^*Vf(x) = a\int_x^1 \int_0^t e^{i\omega s}ds + b\int_x^1 \int_0^t e^{-i\omega s}ds$$

$$= \frac{1}{\omega^2}f(x) + \frac{1}{i\omega}(a-b)x - \frac{1}{\omega^2}\left(ae^{i\omega} + be^{-i\omega}\right) - \frac{1}{i\omega}(a-b)$$

From the second term, we must have $a = b$ and $f$ therefore has the form $2a\cos(\omega x)$. From the third term, since $a \neq 0$, $\cos(i\omega) = 0$. It follows that $f$ is an eigenfunction if and only if $\omega = \frac{2n+1}{2}\pi$. The eigenvalues of $V^*V$ are therefore

$$\lambda_n = \frac{4}{(2n+1)^2\pi^2}.$$

Maximizing over $n$ we see that the largest eigenvalue of $V^*V$, and therefore its operator norm, is $\lambda_0 = \frac{4}{\pi^2}$. We conclude that $\|V\| = \frac{2}{\pi}$.

The operator norm is crucially dependent upon the operator's domain of definition. For example, consider instead $V : L^1[0,1] \to L^1[0,1]$. Then

$$\|Vf\| \leq \int_0^1 \int_0^t |f(s)|dsdt \leq \int_0^1 \int_0^1 |f(s)|dsdt = \|f\|.$$

Therefore $\|V\| \leq 1$. Using the same example we used earlier to rule out compactness of $V$, as $n \to \infty$,

$$\|Vf_n\| = \int_0^1 \int_0^t n\mathbb{1}_{[0,1/n]}(s)dsdt = \int_0^{1/n} ntdt + \int_{1/n}^1 dt = 1 - \frac{1}{2n} \to 1.$$

So in this case $\|V\| = 1$.

## 1. HILBERT-SCHMIDT OPERATORS

Let $L(U)$ denote the space of linear operators on a linear space $U$ over $F$. When $U$ is finite dimensional we can identify $U \otimes U^*$ with $L(U)$ by associating $u \otimes v^* \in U \otimes U^*$ with $T \in L(U)$ such that $T(w) = v^*(w)u$. Let $\mathrm{ev} : U \times U^* \to F$ be the (bilinear) evaluation functional defined by $\mathrm{ev}(u, v^*) = v^*(u)$. By universality of the tensor product this induces a map $\mathrm{tr} : U \otimes U^* \to F$. We identify this map with a map $\mathrm{tr} : L(U) \to F$, which we call the trace.

In infinite dimensions, $U \otimes U^*$ identifies only with finite rank operators. From the preceeding discussion, we can define a Hilbert-Schmidt inner product of finite rank operators $S, T \in L(U)$,

$$\langle S, T\rangle_{HS} = \mathrm{tr}(S^*T).$$

And we define the space of Hilbert-Schmidt operators to be the completion of the finite rank operators with respect to this inner product. Let $\mathrm{ev}_u : U \otimes U^* \to U$ be the evaluation map on finite rank operators defined by $\mathrm{ev}_u(T) = Tu$. This map is uniformly continuous and by universality of completion it induces a map on the Hilbert-Schmidt operators. Furthermore, if $T$ is Hilbert-Schmidt then

$$T(\alpha u + v) = \lim_{n\to\infty} T_n(\alpha u + v) = \alpha \lim_{n\to\infty} T_n u + \lim_{n\to\infty} T_n v = \alpha T u + T v.$$

Limits here are taken in the sense of the topology induced by the Hilbert-Schmidt inner product. We therefore identify Hilbert-Schmidt operators with a subspace of $L(U)$.

Suppose $H$ is a Hilbert space with (possibly uncountable) orthonormal basis $(e_i)_{i\in B}$ and let $u \in H$. If $A$ is a bounded linear operator on $H$ then $A$ is continuous and has an abstract fourier representation

$$Au = A \sum_{i\in B} \langle u, e_i\rangle e_i = \sum_{i\in B} \langle u, e_i\rangle A e_i.$$

The latter sum converges in the sense of the operator topology. Let $e_i^* = \langle e_i, \cdot\rangle_{HS}$ be the dual basis associated with the Hilbert-Schmidt inner product. Then

$$Au = \sum_{i\in B} e_i^*(u) A e_i = \left(\sum_{i\in B} e_i^* \otimes A e_i\right)(u).$$

Here the latter sum converges in the Hilbert-Schmidt topology. It follows that

$$\mathrm{tr}(A) = \mathrm{tr} \sum_{i\in B} e_i^* \otimes A e_i = \sum_{i\in B} \mathrm{tr}(e_i^* \otimes A e_i) = \sum_{i\in B} \langle A e_i, e_i\rangle.$$

We can now compute the Hilbert-Schmidt norm of an operator $A$:

$$\|A\|_{HS}^2 = \langle A, A\rangle_{HS} = \mathrm{tr}(A^*A) = \sum_{i\in B} \langle A^*A e_i, e_i\rangle = \sum_{i\in B} \|A e_i\|^2.$$

Observe that $\|A\|_{HS}^2 < \infty$ and therefore $\|Ae_i\| = 0$ for all but countably many $i \in B$. If we reorder $(e_i)$ as a countable sequence then $(Ae_i)$ as a sequence in $\ell^2$. Furthermore, by

Bessel's inequality

$$\sum_{i=1}^{\infty} |\langle u, e_i \rangle|^2 \leq \|u\|^2 < \infty.$$

Therefore $\langle u, e_i \rangle$ is also a sequence in $\ell^2$. By Cauchy-Schwarz in $\ell^2$,

$$\|Au\| \leq \sum_{i=1}^{\infty} \|\langle u, e_i \rangle A e_i\| = \sum_{i=1}^{\infty} |\langle u, e_i \rangle| \|A e_i\| \leq \left( \sum_{i=1}^{\infty} |\langle u, e_i \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} \|A e_i\|^2 \right)^{\frac{1}{2}}.$$

This sum is finite from preceeding calculations and moreover

$$\|Au\|^2 \leq \|A\|_{HS} \|u\|.$$

It follows that $\|A\|_{op} \leq \|A\|_{HS}$. Therefore Hilbert-Schmidt limits of finite rank operators are operator norm limits of finite rank operators, which are compact. We conclude that Hilbert-Schmidt operators are compact.

Does $L^2(X \times X, \mu \otimes \mu) = L^2(X, \mu) \oplus L^2(X, \mu)$?

Let $X$ be a $\sigma$-finite measure space with $k \in L^2(X \times X)$. We associate $k$ (which we call a kernel) with an integral operator $K$ by the map

$$Ku(x) = \int_X k(x, y) u(y) dy.$$

We will show that $K$ is a Hilbert-Schmidt operator on $L^2(X)$. By Fubini's theorem, $k(x, \cdot) \in L^2(X)$ almost everywhere. If $k(x, \cdot) \in L^2(X)$ and $u \in L^2(X)$ then by Cauchy-Schwarz on $L^2(X)$,

$$\int_X |k(x, y) u(y)| dy \leq \|k(x, \cdot)\| \|u\| < \infty.$$

Therefore $k(x, y) u(y) \in L^1(X)$ and $K$ is defined on $L^2(X)$ (modulo null sets). Another application of Fubini shows that

$$\int_Y |Ku(x)|^2 dx \leq \int_X \|k(x, \cdot)\|^2 \|u\|^2 dx = \|u\|^2 \int_{X \times X} |k(x, y)|^2 dy \otimes dx = \|u\|^2 \|k\|^2 < \infty.$$

It follows that $Ku \in L^2(X)$ and we consider $K$ as a linear operator $K : L^2(X) \to L^2(X)$. Our work further implies that $\|K\| \leq \|k\|$.

Let $i \in B$ index an orthonormal basis $(e_i)$ of $X$. By Bessel's inequality

$$\|K\|_{HS}^2 = \operatorname{tr}(K^* K) = \sum_{i \in B} \|K e_i\|^2 \leq \sum_{i \in B} \sum_{j \in B} |\langle K e_i, e_j \rangle|^2.$$

And by Fubini's theorem again,

$$\langle K e_i, e_j \rangle = \int_X K e_i(x) \bar{e}_j(x) dx = \int_{X \times X} k(x, y) e_i(y) \bar{e_j}(x) dy \otimes dx.$$

Let $u_{ij}(x,y) = e_i(y)\bar{e_j}(x)$ and observe that $(u_{ij})$ forms an orthonormal basis for $L^2(X \times X)$. Then by another application of Bessel's inequality,

$$\|K\|_{HS}^2 \leq \sum_{i\in B}\sum_{j\in B} |\langle Ke_i, e_j\rangle|^2 = \sum_{i,j\in B\times B} |\langle k, u_{ij}\rangle|^2 \leq \|k\|^2.$$

When $L^2(X)$ is separable Bessel's inequality is replaced by Parseval's identity and equality holds. We deduce that $K$ is a Hilbert-Schmidt operator and (for separable spaces) the map $k \mapsto K$ is (Hilbert-Schmidt)-isometric.

Remarkably, any Hilbert-Schmidt operator can be characterized by a kernel. Suppose $K$ is Hilbert-Schmidt. Then $K$ is can be written as a Hilbert-Schmidt limit of finite rank operators $K_n$. For $u_i, v_i \in L^2(X)$, we can write

$$K_n w(x) = \sum_{i=1}^n u_i \otimes v_i(x) = \sum_{i=1}^n \int_X u_i(x)\bar{v}_i(y)w(y)dy = \int_X \sum_{i=1}^n u_i(x)\bar{v}_i(y)w(y)dy.$$

The kernel here is given by

$$k_n(x,y) = \sum_{i=1}^n u_i(x)\bar{v}_i(y).$$

Because $k_n \mapsto K_n$ is a Hilbert-Schmidt isometry and $(K_n)$ converges,

$$\|k_n - k_m\|_{L^2} = \|K_n - K_m\|_{HS} \to 0.$$

Therefore $(k_n)$ is Cauchy and by completness of $L^2(X)$ we deduce that $k_n \to k$ for some $k \in L^2(X)$. This permits us to write

$$Kw(x) = \int_X k(x,y)w(y)dy.$$