A Short Course on

Spatial Vector Algebra

The Easy Way to do Rigid Body Dynamics

Roy Featherstone
Dept. Information Engineering, RSISE
The Australian National University

Spatial vector algebra is a concise vector notation for describing rigid−body velocity, acceleration, inertia, etc., using 6D vectors and tensors.

- fewer quantities
- fewer equations
- less effort
- fewer mistakes

Mathematical Structure

spatial vectors inhabit *two* vector spaces:

M^6 — motion vectors

F^6 — force vectors

with a scalar product defined *between* them

$$\mathbf{m} \cdot \mathbf{f} = \mathit{work}$$

$$\text{“}\cdot\text{”} : \mathsf{M}^6 \times \mathsf{F}^6 \mapsto \mathsf{R}$$

Bases

A coordinate vector $\underline{\mathbf{m}}=[m_1,\ldots,m_6]^T$ represents a motion vector $\mathbf{m}$ in a basis $\{\mathbf{d}_1,\ldots,\mathbf{d}_6\}$ on M^6 if

$$\mathbf{m} = \sum_{i=1}^{6} m_i \, \mathbf{d}_i$$

Likewise, a coordinate vector $\underline{\mathbf{f}}=[f_1,\ldots,f_6]^T$ represents a force vector $\mathbf{f}$ in a basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_6\}$ on F^6 if

$$\mathbf{f} = \sum_{i=1}^{6} f_i \, \mathbf{e}_i$$

Bases

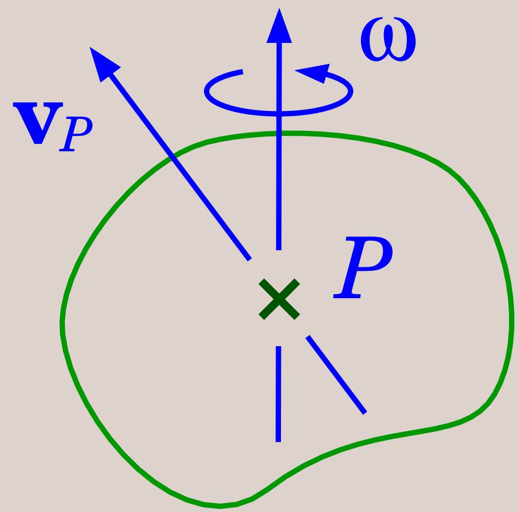
If $\{\mathbf{d}_1, \ldots, \mathbf{d}_6\}$ is an arbitrary basis on M^6 then there exists a unique *reciprocal basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_6\}$ on F^6 satisfying

$$\mathbf{d}_i \cdot \mathbf{e}_j = \begin{cases} 0 : & i \neq j \\ 1 : & i = j \end{cases}$$

With these bases, the scalar product of two coordinate vectors is

$$\mathbf{m} \cdot \mathbf{f} = \underline{\mathbf{m}}^T \, \underline{\mathbf{f}}$$

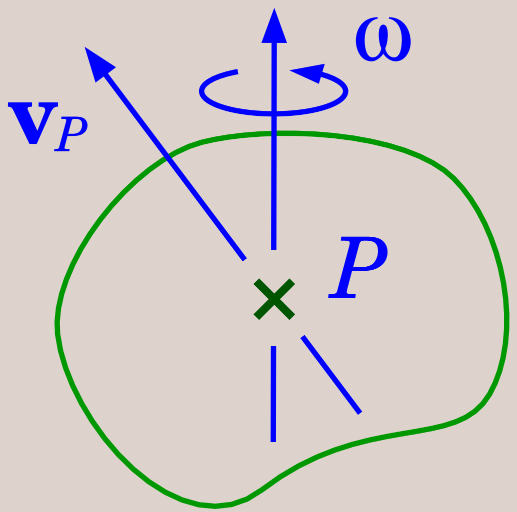
Velocity



The velocity of a rigid body can be described by

- choosing a point, P, in the body
- specifying the linear velocity, $\mathbf{v}_P$, of that point
- specifying the angular velocity, $\boldsymbol{\omega}$, of the body as a whole

Velocity



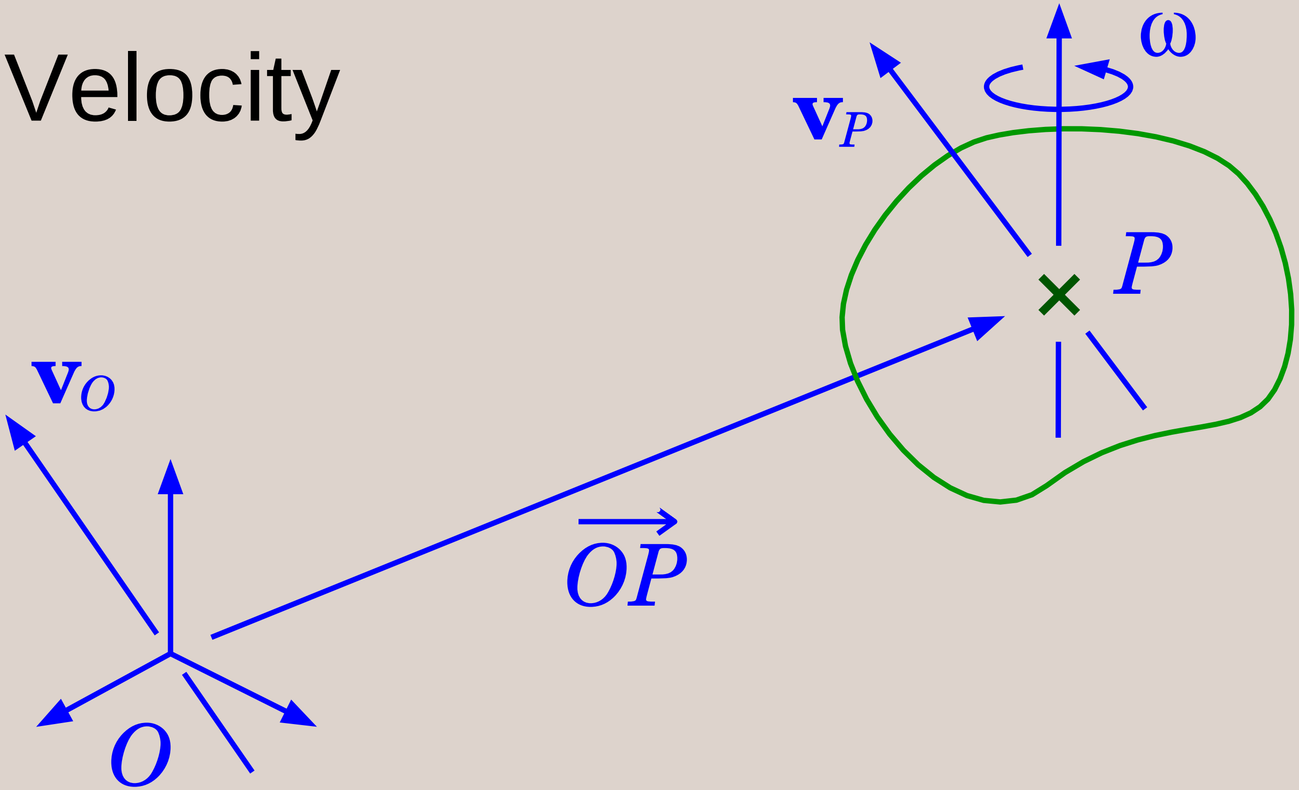
The body is then deemed to be

translating with a linear velocity $\mathbf{v}_P$

while simultaneously

rotating with an angular velocity ω about an axis passing through P

Velocity

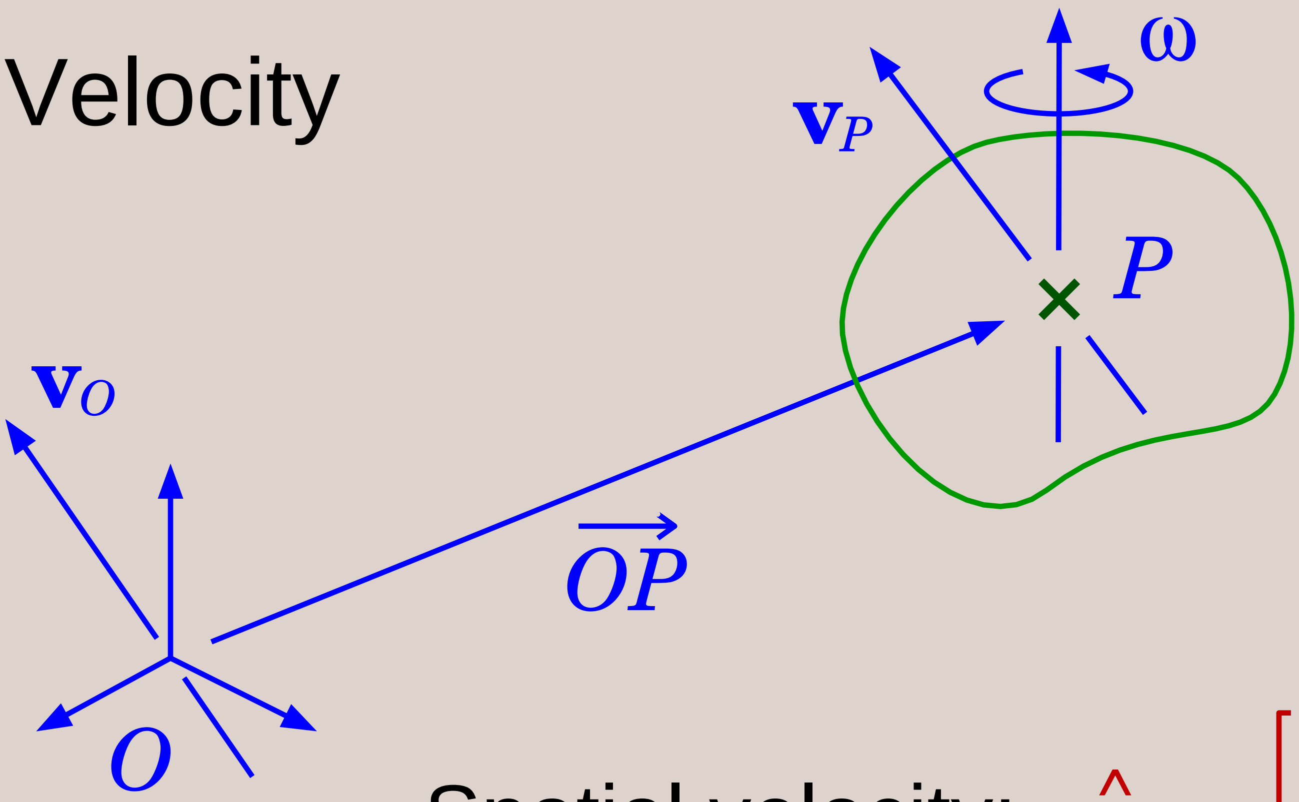


velocity $(\omega, \mathbf{v}_P)$ at P

is equivalent to $(\omega, \mathbf{v}_O)$ at O

where $\mathbf{v}_O = \mathbf{v}_P + \overrightarrow{OP} \times \omega$

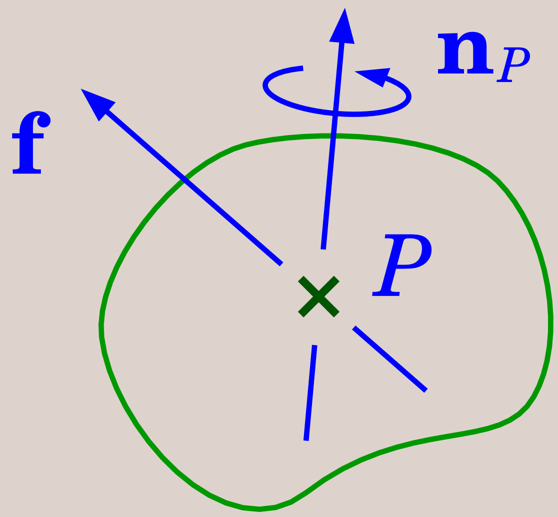
Velocity



Spatial velocity: $\hat{\mathbf{v}}_O = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_O \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_{Ox} \\ v_{Oy} \\ v_{Oz} \end{bmatrix}$

These are the *Plücker coordinates* of $\hat{\mathbf{v}}$ in the frame $Oxyz$

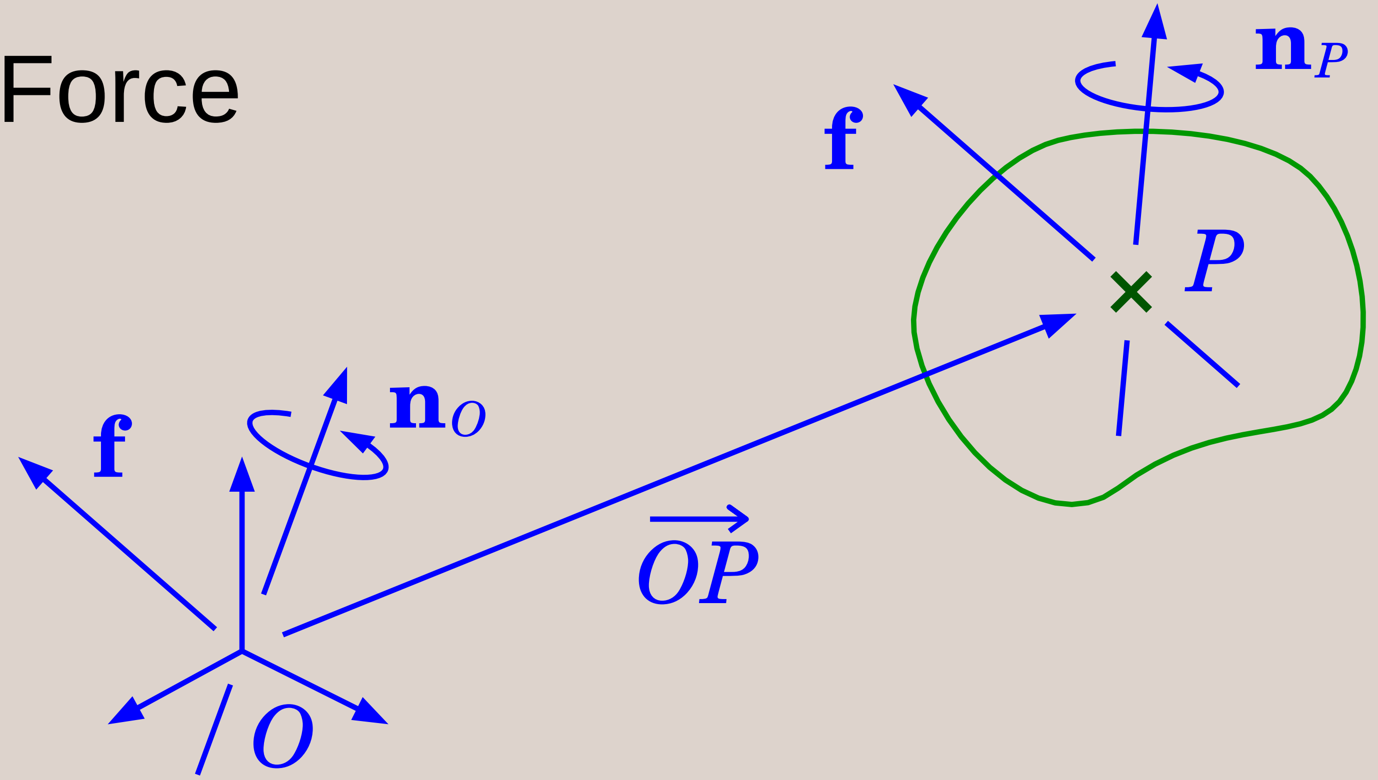
Force



A general force acting on a rigid body is equivalent to the sum of

- a force $\mathbf{f}$ acting along a line passing through a point P, and
- a couple, $\mathbf{n}_P$

Force

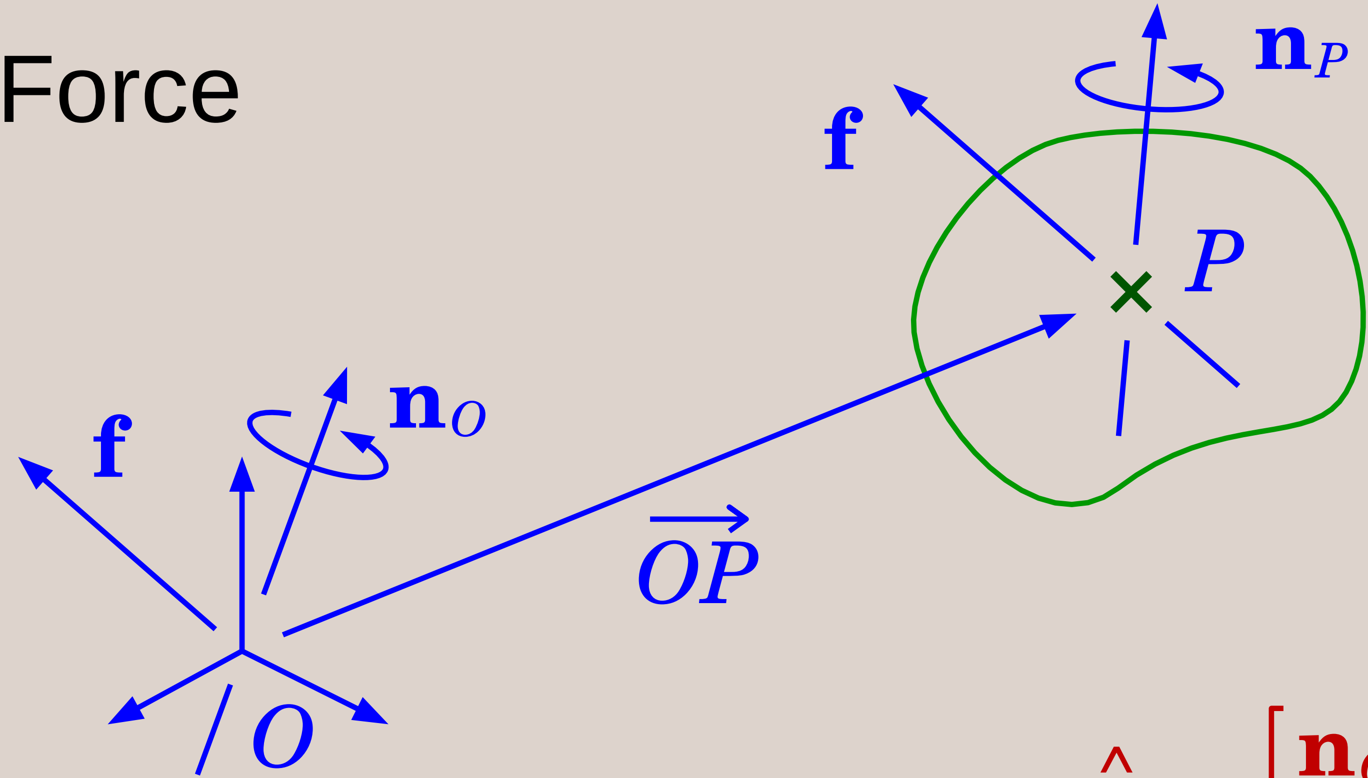


general force $(\mathbf{f}, \mathbf{n}_P)$ at P

is equivalent to $(\mathbf{f}, \mathbf{n}_O)$ at O

where $\mathbf{n}_O = \mathbf{n}_P + \overrightarrow{OP} \times \mathbf{f}$

Force



Spatial force: $\hat{\mathbf{f}}_O = \begin{bmatrix} \mathbf{n}_O \\ \mathbf{f} \end{bmatrix} = \begin{bmatrix} n_{Ox} \\ n_{Oy} \\ n_{Oz} \\ f_x \\ f_y \\ f_z \end{bmatrix}$

These are the *Plücker coordinates* of $\hat{\mathbf{f}}$ in the frame $Oxyz$

Plücker Coordinates

A Cartesian coordinate frame $Oxyz$ defines *twelve* basis vectors:

$\mathbf{d}_{Ox}, \mathbf{d}_{Oy}, \mathbf{d}_{Oz}, \mathbf{d}_x, \mathbf{d}_y, \mathbf{d}_z$:

rotations about the Ox, Oy and Oz axes,
translations in the x, y and z directions

$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z, \mathbf{e}_{Ox}, \mathbf{e}_{Oy}, \mathbf{e}_{Oz}$:

couples in the yz, zx and xy planes, and
forces along the Ox, Oy and Oz axes

If $\hat{\mathbf{v}}_O = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_O \end{bmatrix}$ and $\hat{\mathbf{f}}_O = \begin{bmatrix} \mathbf{n}_O \\ \mathbf{f} \end{bmatrix}$ are the Plücker coordinates of $\hat{\mathbf{v}}$ and $\hat{\mathbf{f}}$ in $Oxyz$, then

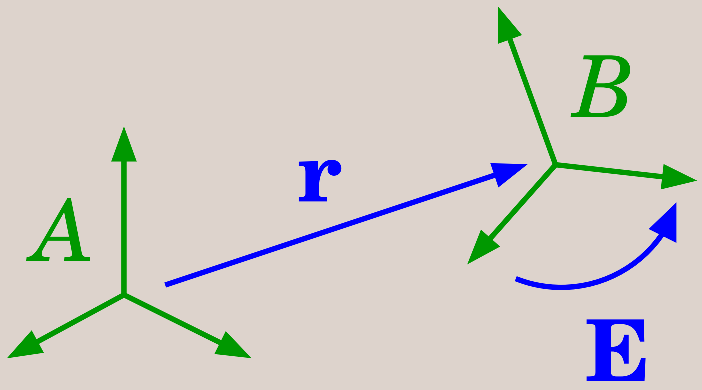
$$\hat{\mathbf{v}} = \omega_x\, \mathbf{d}_{Ox} + \omega_y\, \mathbf{d}_{Oy} + \omega_z\, \mathbf{d}_{Oz} + \\ + v_{Ox}\, \mathbf{d}_x + v_{Oy}\, \mathbf{d}_y + v_{Oz}\, \mathbf{d}_z$$

$$\hat{\mathbf{f}} = n_{Ox}\, \mathbf{e}_x + n_{Oy}\, \mathbf{e}_y + n_{Oz}\, \mathbf{e}_z + \\ + f_x\, \mathbf{e}_{Ox} + f_y\, \mathbf{e}_{Oy} + f_z\, \mathbf{e}_{Oz}$$

and

$$\hat{\mathbf{v}} \cdot \hat{\mathbf{f}} = \hat{\mathbf{v}}_O^T\, \hat{\mathbf{f}}_O$$

Coordinate Transforms



transform from A to B
for motion vectors:

$$ {}^B\mathbf{X}_A = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \tilde{\mathbf{r}}^T & \mathbf{1} \end{bmatrix} $$

where

$$ \tilde{\mathbf{r}} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} $$

corresponding transform
for force vectors:

$$ {}^B\mathbf{X}_A^F = ({}^B\mathbf{X}_A)^{-T} $$

Basic Operations with Spatial Vectors

- Composition of velocities

If $\mathbf{v}_A$ = velocity of body A

$\mathbf{v}_B$ = velocity of body B

$\mathbf{v}_{BA}$ = relative velocity of B w.r.t. A

Then $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA}$

- Scalar multiples

If $\mathbf{s}$ and $\dot{q}$ are a joint motion axis and velocity variable, then the joint velocity is $\mathbf{v}_J = \mathbf{s}\,\dot{q}$

- Composition of forces

  If forces $\mathbf{f}_1$ and $\mathbf{f}_2$ both act on the same body then their resultant is

  $$\mathbf{f}_{tot} = \mathbf{f}_1 + \mathbf{f}_2$$

- Action and reaction

  If body A exerts a force $\mathbf{f}$ on body B, then body B exerts a force $-\mathbf{f}$ on body A (Newton's 3rd law)

Now try question set A

Spatial Cross Products

There are *two* cross product operations: one for motion vectors and one for forces

$$\hat{\mathbf{v}}_O \times \hat{\mathbf{m}}_O = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_O \end{bmatrix} \times \begin{bmatrix} \mathbf{m} \\ \mathbf{m}_O \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega} \times \mathbf{m} \\ \boldsymbol{\omega} \times \mathbf{m}_O + \mathbf{v}_O \times \mathbf{m} \end{bmatrix}$$

$$\hat{\mathbf{v}}_O \times \hat{\mathbf{f}}_O = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_O \end{bmatrix} \times \begin{bmatrix} \mathbf{n}_O \\ \mathbf{f} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega} \times \mathbf{n}_O + \mathbf{v}_O \times \mathbf{f} \\ \boldsymbol{\omega} \times \mathbf{f} \end{bmatrix}$$

where $\hat{\mathbf{v}}_O$ and $\hat{\mathbf{m}}_O$ are motion vectors, and $\hat{\mathbf{f}}_O$ is a force.

Differentiation

- The derivative of a spatial vector is itself a spatial vector
- in general, $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{s} = \lim\limits_{\delta t \to 0} \dfrac{\mathbf{s}(t+\delta t)-\mathbf{s}(t)}{\delta t}$
- The derivative of a spatial vector that is fixed in a moving body with velocity $\mathbf{v}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{s} = \mathbf{v}\times\mathbf{s}$$

Differentiation in Moving Coordinates

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{s}\right]_O = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{s}_O + \mathbf{v}_O \times \mathbf{s}_O$$

velocity of coordinate frame

componentwise derivative of coordinate vector

coordinate vector representing $\mathrm{d}\mathbf{s}/\mathrm{d}t$

Acceleration

. . . is the rate of change of velocity:

$$\hat{\mathbf{a}} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{v}} = \begin{bmatrix} \dot{\boldsymbol{\omega}} \\ \dot{\mathbf{v}}_O \end{bmatrix}$$

but this is *not* the linear acceleration of any point in the body!

Classical acceleration is

$$\begin{bmatrix} \dot{\boldsymbol{\omega}} \\ \dot{\mathbf{v}}'_O \end{bmatrix}$$

where $\dot{\mathbf{v}}'_O$ is the derivative of $\mathbf{v}_O$ taking O to be fixed in the body

Spatial acceleration is

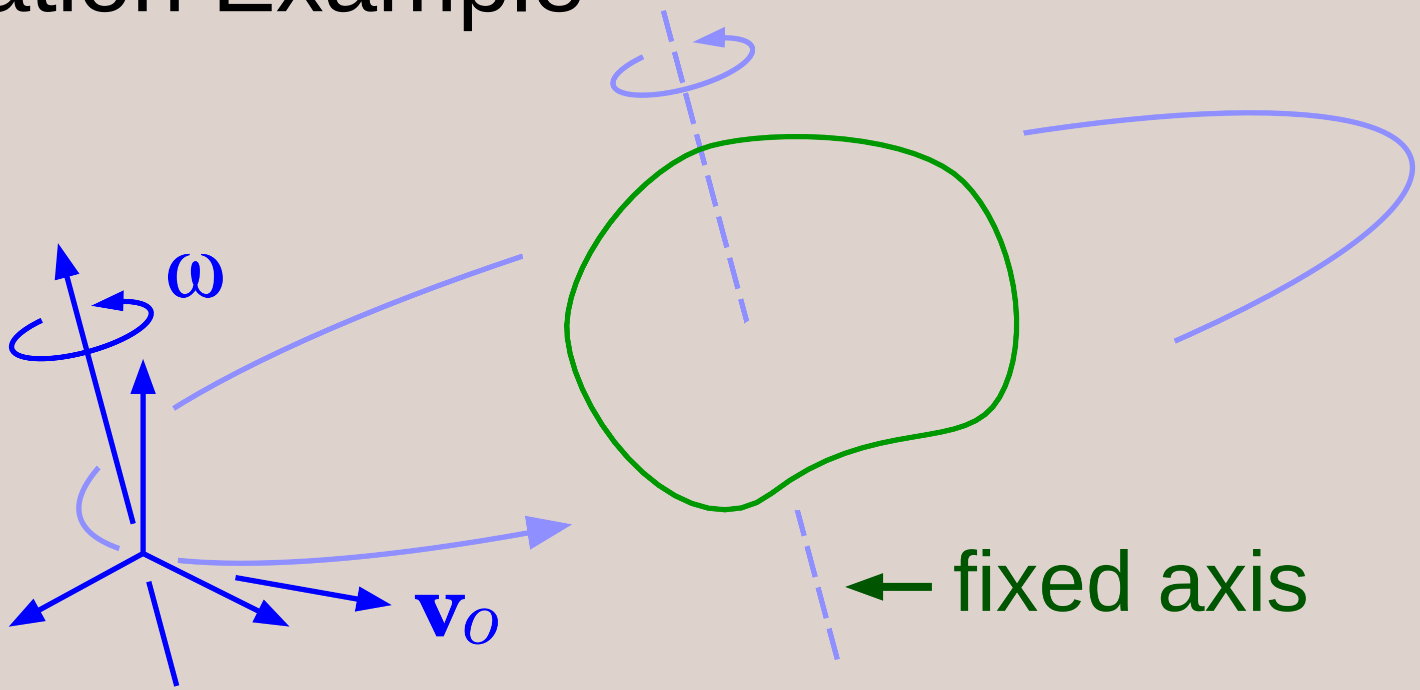
$$\begin{bmatrix} \dot{\boldsymbol{\omega}} \\ \dot{\mathbf{v}}_O \end{bmatrix}$$

where $\dot{\mathbf{v}}_O$ is the derivative of $\mathbf{v}_O$ taking O to be fixed in space

What's the difference?

Spatial acceleration is a vector.

Acceleration Example



A body rotates with constant angular velocity $\boldsymbol{\omega}$, so its spatial velocity is constant, so its spatial acceleration is zero; but almost every body−fixed point is accelerating.

Basic Operations with Accelerations

- Composition

If $\mathbf{a}_A$ = acceleration of body A

$\mathbf{a}_B$ = acceleration of body B

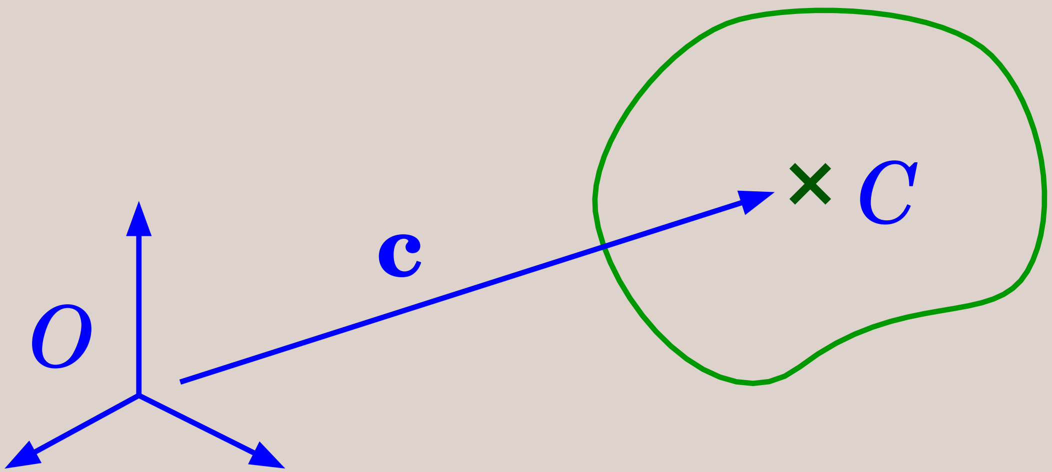
$\mathbf{a}_{BA}$ = acceleration of B w.r.t. A

Then $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA}$

Look, no Coriolis term!

Now try question set B

Rigid Body Inertia



mass: m

CoM: C

inertia at CoM: $\mathbf{I}_C$

spatial inertia tensor: $$\hat{\mathbf{I}}_O = \begin{bmatrix} \mathbf{I}_O & m\,\tilde{\mathbf{c}} \\ m\,\tilde{\mathbf{c}}^T & m\,\mathbf{1} \end{bmatrix}$$

where $\mathbf{I}_O = \mathbf{I}_C - m\,\tilde{\mathbf{c}}\,\tilde{\mathbf{c}}$

Basic Operations with Inertias

- Composition

  If two bodies with inertias $\mathbf{I}_A$ and $\mathbf{I}_B$ are joined together then the inertia of the composite body is

$$\mathbf{I}_{tot} = \mathbf{I}_A + \mathbf{I}_B$$

- Coordinate transformation formula

$$\mathbf{I}_B = {}^B\mathbf{X}_A^F \, \mathbf{I}_A \, {}^A\mathbf{X}_B = ({}^A\mathbf{X}_B)^T \, \mathbf{I}_A \, {}^A\mathbf{X}_B$$

Equation of Motion

$$\mathbf{f} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}\,\mathbf{v}) = \mathbf{I}\,\mathbf{a} + \mathbf{v} \times \mathbf{I}\,\mathbf{v}$$

$\mathbf{f}$ = net force acting on a rigid body

$\mathbf{I}$ = inertia of rigid body

$\mathbf{v}$ = velocity of rigid body

$\mathbf{I}\,\mathbf{v}$ = momentum of rigid body

$\mathbf{a}$ = acceleration of rigid body

Motion Constraints

If a rigid body's motion is constrained, then its velocity is an element of a subspace, $S \subset \mathrm{M}^6$, called the *motion subspace*

degree of (motion) freedom: $\dim(S)$

degree of constraint: $6 - \dim(S)$

S can vary with time

Matrix Representation

Let $\mathbf{S}$ be any $6 \times \dim(S)$ matrix such that $\mathrm{range}(\mathbf{S}) = S$, then

- the columns of $\mathbf{S}$ define a basis on S
- any vector $\mathbf{v} \in S$ can be expressed in the form $\mathbf{v} = \mathbf{S}\,\alpha$, where α is a $\dim(S) \times 1$ vector of coordinates

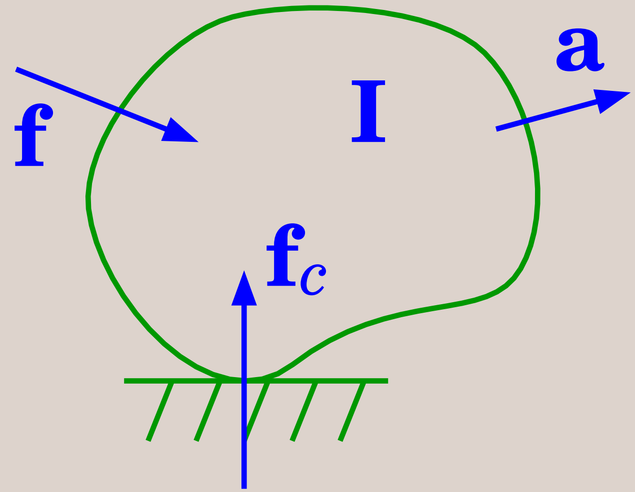
Constraint Forces

- motion constraints are enforced by constraint forces
- constraint forces do no work: if $\mathbf{f}_c$ is the constraint force, and $\mathbf{S}$ the motion subspace matrix, then

$$\mathbf{S}^T \mathbf{f}_c = \mathbf{0}$$

- constraint forces occupy a subspace $T \subset \mathsf{F}^6$ satisfying $\dim(S) + \dim(T) = 6$ and $\mathbf{S}^T \mathbf{T} = \mathbf{0}$

Constrained Motion

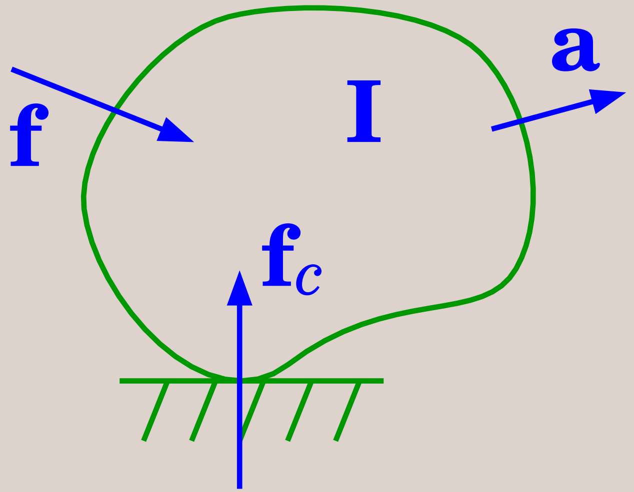


A force, $\mathbf{f}$, is applied to a body with inertia $\mathbf{I}$ that is constrained to a subspace $S = \mathrm{Range}(\mathbf{S})$ of M^6. Assuming the body is initially at rest, what is its acceleration?

velocity	$\mathbf{v} = \mathbf{S}\,\alpha = \mathbf{0}$
acceleration	$\mathbf{a} = \mathbf{S}\,\dot{\alpha} + \dot{\mathbf{S}}\,\alpha = \mathbf{S}\,\dot{\alpha}$
constraint force	$\mathbf{S}^T \mathbf{f}_c = \mathbf{0}$

equation of motion

$$\mathbf{f} + \mathbf{f}_c = \mathbf{I}\,\mathbf{a} + \mathbf{v} \times \mathbf{I}\,\mathbf{v}$$
$$= \mathbf{I}\,\mathbf{a}$$



solution

$$\mathbf{f} + \mathbf{f}_c = \mathbf{I}\,\mathbf{S}\,\dot{\alpha}$$

$$\mathbf{S}^T\mathbf{f} = \mathbf{S}^T\mathbf{I}\,\mathbf{S}\,\dot{\alpha}$$

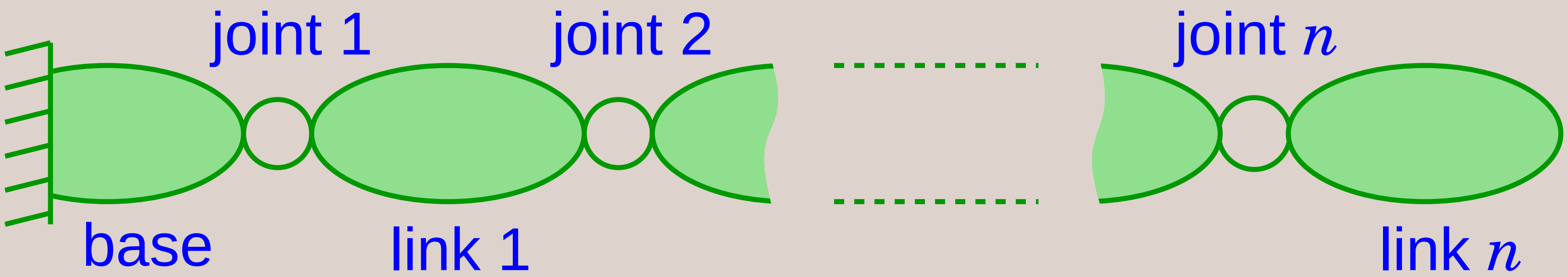
$$\dot{\alpha} = (\mathbf{S}^T\mathbf{I}\,\mathbf{S})^{-1}\mathbf{S}^T\mathbf{f}$$

$$\mathbf{a} = \underbrace{\mathbf{S}\,(\mathbf{S}^T\mathbf{I}\,\mathbf{S})^{-1}\mathbf{S}^T}_{\text{apparent inverse inertia}}\,\mathbf{f}$$

apparent
inverse inertia

Now try question set C

Inverse Dynamics



$\dot{q}_i$, $\ddot{q}_i$, $\mathbf{s}_i$	joint velocity, acceleration & axis
$\mathbf{v}_i$, $\mathbf{a}_i$	link velocity and acceleration
$\mathbf{f}_i$	force transmitted from link $i{-}1$ to i
τ_i	joint force variable
$\mathbf{I}_i$	link inertia

- velocity of link i is the velocity of link $i{-}1$ plus the velocity across joint i

$$\mathbf{v}_i = \mathbf{v}_{i-1} + \mathbf{s}_i \, \dot{q}_i$$

- acceleration is the derivative of velocity

$$\mathbf{a}_i = \mathbf{a}_{i-1} + \dot{\mathbf{s}}_i \, \dot{q}_i + \mathbf{s}_i \, \ddot{q}_i$$

- equation of motion

$$\mathbf{f}_i - \mathbf{f}_{i+1} = \mathbf{I}_i \, \mathbf{a}_i + \mathbf{v}_i \times \mathbf{I}_i \, \mathbf{v}_i$$

- active joint force

$$\tau_i = \mathbf{s}_i^T \, \mathbf{f}_i$$

The Recursive Newton−Euler Algorithm

(Calculate the joint torques τ_i that will produce the desired joint accelerations $\ddot{q}_i$.)

$$\mathbf{v}_i = \mathbf{v}_{i-1} + \mathbf{s}_i\,\dot{q}_i \qquad (\mathbf{v}_0 = \mathbf{0})$$

$$\mathbf{a}_i = \mathbf{a}_{i-1} + \dot{\mathbf{s}}_i\,\dot{q}_i + \mathbf{s}_i\,\ddot{q}_i \qquad (\mathbf{a}_0 = \mathbf{0})$$

$$\mathbf{f}_i = \mathbf{f}_{i+1} + \mathbf{I}_i\,\mathbf{a}_i + \mathbf{v}_i \times \mathbf{I}_i\,\mathbf{v}_i \qquad (\mathbf{f}_{n+1} = \mathbf{f}_{ee})$$

$$\tau_i = \mathbf{s}_i^T\,\mathbf{f}_i$$