LOWER BOUNDS ON UNIVERSAL TRAVERSAL SEQUENCES FOR CYCLES AND OTHER LOW DEGREE GRAPHS*

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Abstract. Universal traversal sequences for cycles require length $\Omega(n^{1.29})$, improving the previous bound of $\Omega(n \log n)$. For $d \geq 3$, universal traversal sequences for d-regular graphs require length $\Omega(d^{0.71} n^{2.29})$. For constant d, the best previous bound was $\Omega(n^2 \log n)$.

Key words. Universal traversal sequence, reflecting sequence, circumnavigation sequence, cycle, lower bound, graph traversal, time-space tradeoff

1. Reflecting and Universal Traversal Sequences. Universal traversal sequences were introduced by Cook (see Aleliunas *et al.* [2]) as a particularly simple method for traversing graphs. Universal traversal sequences are defined as follows.

For $2 \leq d < n$, let $\mathcal{G}(d,n)$ be the set of all connected, d-regular, n-vertex, edge-labeled, undirected graphs $G = (V,E)$. For this definition, edges are labeled as follows. For every edge $\{u,v\} \in E$ there are two labels $l_{u,v}$ and $l_{v,u}$ with the property that, for every $u \in V$, $\{l_{u,v} \mid \{u,v\} \in E\} = \{0,1,\ldots,d-1\}$. For such labeled graphs, a string over $\{0,1,\ldots,d-1\}$ can be thought of as a sequence of edge traversal commands. In particular, any $U = U_1 U_2 \ldots U_k \in \{0,1,\ldots,d-1\}^*$ and $v_0 \in V$ determine a unique sequence $(v_0, v_1, \ldots, v_k) \in V^{k+1}$ such that $l_{v_{i-1},v_i} = U_i$, for all $i \in \{1,2,\ldots,k\}$. Such a sequence U is said to *traverse* G starting at v_0 if and only if every vertex in G appears at least once in the sequence $v_0, v_1, \ldots, v_k$. U is a *universal traversal sequence* for $\mathcal{G}(d,n)$ if and only if U traverses each $G \in \mathcal{G}(d,n)$ starting at any vertex in G. $U(d,n)$ denotes the length of a shortest universal traversal sequence for $\mathcal{G}(d,n)$. To avoid trivialities, define $U(d,n) = U(d,n+1)$ in case $\mathcal{G}(d,n)$ is empty, which occurs exactly when both d and n are odd (see [5, Proposition 1]).

Good bounds on $U(d,n)$ translate into good bounds on the time required by certain simple undirected graph traversal algorithms running in very limited space. In particular, determining good lower bounds on $U(d,n)$ is a prerequisite to proving time-space tradeoffs for traversing undirected graphs. (See Borodin, Ruzzo, and Tompa [5] for more detailed discussion.) Proving such lower bounds is the emphasis of this paper.

The current best upper and lower bounds on $U(d,n)$ are summarized in Table 1. (See Borodin, Ruzzo, and Tompa [5] for more background.) Prior to the current work, the best lower bound for $d = 2$ was

$$U(2,n) = \Omega(n \log n),$$

due to Bar-Noy *et al.* [4]. This bound is improved in Corollary 3.5 to

$$U(2,n) = \Omega(n^{\log_4 6}) = \Omega(n^{1.29}).$$

For $3 \leq d \leq n/3 - 2$, the best previous lower bound was

$$U(d,n) = \Omega(dn^2 \log \frac{n}{d} + d^2 n^2),$$

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TABLE 1
Best Known Bounds on Length of Universal Traversal Sequences

Bound	*Relevant Range*	*Source*
$U(d,n) = O(n^3)$	$d = 2$	Aleliunas [1]
$U(d,n) = O(dn^3 \log n)$	$3 \le d \le n/2 - 1$	Kahn *et al.* [7]
$U(d,n) = O(n^3 \log n)$	$n/2 - 1 < d$	Chandra *et al.* [6]
$U(d,n) = \Omega(n^{1.29})$	$d = 2$	This paper
$U(d,n) = \Omega(d^{0.71} n^{2.29})$	$3 \le d < n^{\log_6 1.5}$	This paper
$U(d,n) = \Omega(d^2 n^2)$	$n^{\log_6 1.5} \le d \le n/3 - 2$	Borodin *et al.* [5]
$U(d,n) = \Omega(n^2)$	$n/3 - 2 < d$	Alon *et al.* [3]

due to Borodin, Ruzzo, and Tompa [5]. This bound is improved in Corollary 3.6 to

$$U(d,n) = \Omega(d^{2-\log_4 6} n^{1+\log_4 6} + d^2 n^2) = \Omega(d^{0.71} n^{2.29} + d^2 n^2).$$

The value of d at which the second term begins to dominate is $d = n^{\log_6 1.5} \approx n^{0.23}$.

By a clever extension of the techniques in this paper, Coppersmith [personal communication] has improved these bounds to $U(2,n) = \Omega(n^{1.33})$ and $U(d,n) = \Omega(d^{0.67} n^{2.33})$ for $d \ge 3$.

The method underlying the new lower bounds depends on analyzing traversals of "labeled chains". A *labeled chain of length* n is an undirected graph $G = (V, E)$, where $V = \{0, 1, \ldots, n\}$ and $E = \{\{i, i+1\} \mid 0 \le i < n\}$, with edge labels as follows. Every edge $\{i, i+1\}$ has two labels $l_{i,i+1}$ and $l_{i+1,i}$, each a nonempty subset of $\{0, 1\}$, restricted as follows:

1. $l_{0,1} = l_{n,n-1} = \{0, 1\}$, and
2. one of $l_{i,i-1}$ and $l_{i,i+1}$ is $\{0\}$ and the other is $\{1\}$, for all $0 < i < n$.

Let $\mathcal{L}(n)$ be the set of labeled chains of length n. Thus, $|\mathcal{L}(n)| = 2^{n-1}$. A labeled chain G will often be identified with the string $\beta = \beta_1 \beta_2 \cdots \beta_{n-1} \in \{0,1\}^{n-1}$, where $l_{i,i+1} = \{\beta_i\}$ for $0 < i < n$. The string β is called the *label* of G.

Given a labeled chain G of length n, a string $U \in \{0,1\}^*$ can be considered as a sequence of edge traversal commands starting at vertex 0. In particular, if $U = U_1 U_2 \cdots U_{|U|}$, where $U_i \in \{0, 1\}$ for $1 \le i \le |U|$, U determines a unique sequence $(v_0 = 0, v_1, v_2, \ldots, v_{|U|}) \in \{0, 1, \ldots, n\}^{|U|+1}$ such that $U_i \in l_{v_{i-1}, v_i}$, for all $1 \le i \le |U|$. Such a sequence U is said to *reflect* t *times* on G if and only if there exist $0 < j_1 < j_2 < \cdots < j_t \le |U|$ such that $v_{j_{2k-1}} = n$ for all $1 \le k \le \lceil t/2 \rceil$ and $v_{j_{2k}} = 0$ for all $1 \le k \le \lfloor t/2 \rfloor$. U is a *t-reflecting sequence* for $\mathcal{L}(n)$ if and only if U reflects t times on each $G \in \mathcal{L}(n)$. $R(t,n)$ denotes the length of a shortest t-reflecting sequence for $\mathcal{L}(n)$.

The remainder of this section demonstrates how lower bounds on $R(t,n)$ imply lower bounds on the lengths of universal traversal sequences. Sections 2 and 3 establish the lower bound $R(t,n) \ge t n^{\log_4 6} > t n^{1.29}$. When combined with the reductions of Theorem 1.1 and Corollary 1.4 below, this yields lower bounds on the length of universal traversal sequences.

For $\beta \in \{0,1\}^*$, define $\bar{\beta}$ to be the string that results from reversing β and then complementing its bits. For instance, if $\beta = 00010$, then $\bar{\beta} = 10111$.

The first reduction shows how a lower bound on $R(1,n)$ implies a lower bound on the length of universal traversal sequences on cycles.

THEOREM 1.1. *For any positive integer n, $U(2,2n) \geq R(1,n)$.*

Proof. Let U be a universal traversal sequence for $\mathcal{G}(2,2n)$. It will be shown that U is a 1-reflecting sequence for $\mathcal{L}(n)$. Choose any labeled chain $G_\beta \in \mathcal{L}(n)$ with label $\beta \in \{0,1\}^{n-1}$. Construct a $2n$-cycle C_β whose clockwise label from its start vertex is $0\beta 0\bar{\beta}$. By induction on $|U|$, it is straightforward to show that the traversal of C_β according to U terminates at a vertex of distance j from the start vertex if and only if the traversal of G_β according to U terminates at vertex j. Since U is universal for $\mathcal{G}(2,2n)$, the vertex at distance n from the start vertex is reached during the traversal of C_β according to U. Hence, U reflects at least 1 time on G_β. □

Corollary 1.4 below demonstrates that a lower bound on the length of general reflecting sequences implies a lower bound on $U(d,n)$ for $d \geq 3$ as well. The key to relating reflecting sequences to universal traversal sequences of degree $d \geq 3$ lies in the notion of "circumnavigation sequences", introduced by Borodin, Ruzzo, and Tompa [5]. For any labeled cycle $C \in \mathcal{G}(2,n)$, a string over $\{0,1\}$ can be interpreted as a traversal sequence. In particular, any $U \in \{0,1\}^*$ and start vertex v_0 of C determine a unique sequence $(v_0, v_1, \ldots, v_{|U|})$ of vertices traversed by U. Such a sequence U is said to circumnavigate C t times starting at v_0 if there are at least t times at which the sequence returns to v_0 moving in the same direction in which it last left v_0. More precisely, *U circumnavigates C t times* if and only if there exist $0 \leq i_1 < i_2 \leq i_3 < i_4 \leq \cdots \leq i_{2t-1} < i_{2t} \leq |U|$ such that

1. $v_0 = v_{i_1} = v_{i_2} = \cdots = v_{i_{2t}}$,
2. $v_l \neq v_0$ for all $i_{2j-1} < l < i_{2j}$ and $1 \leq j \leq t$, and
3. $v_{i_{2j-1}+1} \neq v_{i_{2j}-1}$, for all $1 \leq j \leq t$.

U is a *t-circumnavigation sequence* for $\mathcal{G}(2,n)$ if and only if U circumnavigates each $C \in \mathcal{G}(2,n)$ t times starting at any vertex in C. $C(t,n)$ denotes the length of a shortest t-circumnavigation sequence for $\mathcal{G}(2,n)$.

The next theorem, due to Borodin, Ruzzo, and Tompa [5], relates universal traversal sequences of higher degree graphs to circumnavigations of cycles.

THEOREM 1.2 ([5, SECTION 4.1]). *Let $d \geq 3$ be an integer, and n be a multiple of $8(d-1)$. Then*

$$U(d,n) \geq \frac{d}{2} C\left(\frac{(d-2)n}{4} + 2, \frac{n}{8(d-1)}\right).$$

Using the same reduction as in the proof of Theorem 1.1, it is straightforward to relate reflecting sequences to circumnavigation sequences:

THEOREM 1.3. *For any positive integers t and n, $C(t,2n) \geq R(2t,n)$.*

Combining Theorems 1.2 and 1.3 gives the desired reduction:

COROLLARY 1.4. *Let $d \geq 3$ be an integer, and n be a multiple of $16(d-1)$. Then*

$$U(d,n) \geq \frac{d}{2} R\left(\frac{(d-2)n}{2} + 4, \frac{n}{16(d-1)}\right).$$

2. A Lower Bound on $R(t,n)$.

THEOREM 2.1. *For any positive integers t, m, and n,*

$$R(t,mn) \geq R(R(t,m),n).$$

Proof. Let $U = U_1 U_2 \cdots U_{|U|}$ be a t-reflecting sequence for $\mathcal{L}(mn)$. It will be shown that U is an $R(t,m)$-reflecting sequence for $\mathcal{L}(n)$. Suppose this is not the case.

Then there is a chain $G_\beta \in \mathcal{L}(n)$ with label $\beta \in \{0,1\}^{n-1}$ such that U reflects s times on G_β but not $s+1$ times, where $s < R(t,m)$.

Consider a traversal of G_β according to U, and let $v_0 = 0, v_1, \ldots, v_{|U|}$ be the sequence of vertices visited during this traversal. The traversal will be said to be at vertex v *at time* i if and only if $v_i = v$. Let $j_0 = 0 \le i_1 < j_1 \le i_2 < j_2 \le \cdots \le i_s < j_s \le |U|$ satisfy the following:

1. for all $1 \le k \le \lceil s/2 \rceil$, i_{2k-1} is the last time the traversal is at vertex 0 before it next reaches vertex n, which occurs at time j_{2k-1}, and
2. for all $1 \le k \le \lfloor s/2 \rfloor$, i_{2k} is the last time the traversal is at vertex n before it next reaches vertex 0, which occurs at time j_{2k}.

Let $U' = U_{i_1+1} U_{i_2+1} \cdots U_{i_s+1}$. Since $s < R(t,m)$, U' is not a t-reflecting sequence for $\mathcal{L}(m)$. That is, there is a chain $G_\alpha \in \mathcal{L}(m)$ with label $\alpha = \alpha_1 \alpha_2 \cdots \alpha_{m-1} \in \{0,1\}^{m-1}$ such that U' does not reflect t times on G_α.

Now construct $G \in \mathcal{L}(mn)$ with label

$$\beta \alpha_1 \bar{\beta} \alpha_2 \beta \alpha_3 \bar{\beta} \alpha_4 \cdots \alpha_{m-1} \tilde{\beta},$$

where $\tilde{\beta} = \beta$ if m is odd and $\tilde{\beta} = \bar{\beta}$ if m is even. The contradiction will result because G is constructed so that U "behaves the same" on G as U' does on G_α, yet U reflects t times on G (since U is a t-reflecting sequence for $\mathcal{L}(mn)$) but U' does not on G_α.

EXAMPLE: Before continuing the proof, the construction will be illustrated by showing that

$$U = 11100100010011100100011011000100011011100100010011000100 0100$$

is not a 2-reflecting sequence for $\mathcal{L}(12)$. In this example, $m = 3$, $n = 4$, and $t = 2$. For the first step, U is not a 9-reflecting sequence for $\mathcal{L}(4)$. For instance, U reflects only 8 times on the chain with label $\beta = 011$. For this choice of β, the subsequence U' at which each of the 8 reflections begins is underlined below:

$$U = 11100100010\underline{0}11\underline{1}00100011\underline{0}11\underline{0}0010001\underline{1}011\underline{1}00100010\underline{0}11\underline{0}001000100$$

That is, $U' = 01101100$. Now U' is not a 2-reflecting sequence for $\mathcal{L}(3)$. For instance, U' reflects only 1 time on the chain with label $\alpha = 01$. Hence, U itself reflects only 1 time on the chain with label

$$\overbrace{011}^{\beta}\,0\,\overbrace{001}^{\bar{\beta}}\,1\,\overbrace{011}^{\beta}.$$

The example completed, the proof of Theorem 2.1 will now be continued. The phrase "behaves the same" more precisely means that the traversal of G according to U is at vertex hn at time j_k if and only if the traversal of G_α according to U' is at vertex h at time k, for all $0 \le k \le s$. This will be proved by induction on k. Once it has been established, it implies that the number of reflections of U on G and U' on G_α is the same (since no additional reflections on G can be completed between times j_s and $|U|$), which completes the contradiction.

BASIS ($k = 0$): At time $j_0 = 0$ of the traversal according to U on G, the traversal is at vertex $0 = 0 \cdot n$. At time $k = 0$ of the traversal according to U' on G_α, the traversal is also at vertex 0.

INDUCTION ($k > 0$): Suppose that the traversal of G_α according to U' is at vertex h at time $k-1$, and the traversal of G according to U is at vertex hn at time j_{k-1}.

Assume that h is even, the case when h is odd being dual. Then k is odd, since on G_α the traversal is at an even numbered vertex at time $k-1$. Now between times j_{k-1} and i_k of a traversal according to U on G_β, the traversal starts and ends at vertex 0 without reaching vertex n. Therefore, between times j_{k-1} and i_k of the traversal according to U on G, the traversal starts and ends at vertex hn without reaching either vertex $(h-1)n$ or vertex $(h+1)n$. This is because the only bit from α that affects the traversal during this time interval is α_h, the label at vertex hn, and the traversal is even independent of α_h, except for reflection about vertex hn.

At this point, consider the next step of U' on G_α in conjunction with the next step of U on G. In both cases the label at the current vertex v is $l_{v,v+1} = \{\alpha_h\}$ (unless $h \in \{0, m\}$) and the next bit of the reflecting sequence is U_{i_k+1}. Thus, both traversals move in the same direction in the next step (even if $h \in \{0, m\}$). Assume without loss of generality that both move to the next greater vertex $v+1$. It suffices to prove, then, that the traversal of G according to U is at vertex $(h+1)n$ at time j_k.

Now between times i_k and j_k of a traversal according to U on G_β, the traversal starts at vertex 0 and ends at vertex n, without repeating either. Therefore, between times i_k and j_k of the traversal according to U on G, the traversal starts at vertex hn and ends at vertex $(h+1)n$. This is because, by assumption, its first step is to vertex $hn+1$, after which no bit from α affects it during this time interval. □

COROLLARY 2.2. *Let c and r be positive integers such that, for all positive integers t, $R(t,c) \geq rt$. Then if t is any positive integer and $n = c^m$ for some nonnegative integer m,*

$$R(t,n) \geq tn^{\log_c r}.$$

Proof. It is equivalent to prove that $R(t,c^m) \geq tr^m$, which will be done by induction on m.

BASIS ($m = 0$): $R(t,1) = t$.

INDUCTION ($m > 0$): Assume by the induction hypothesis that $R(t,c^{m-1}) \geq tr^{m-1}$. By Theorem 2.1,

$$\begin{aligned} R(t,c^m) &\geq R(R(t,c^{m-1}),c) \\ &\geq rR(t,c^{m-1}) \\ &\geq tr^m. \end{aligned}$$

□

COROLLARY 2.3. *Let c and r be positive integers such that, for all positive integers t, $R(t,c) \geq rt$. Then $U(2,n) = \Omega(n^{\log_c r})$.*

Proof. From Theorem 1.1 and Corollary 2.2, $U(2,2n) \geq n^{\log_c r}$ whenever n is an integral power of c. The Ω bound follows since $U(2,n)$ is monotone nondecreasing in n. □

COROLLARY 2.4. *Let c and r be positive integers such that, for all positive integers t, $R(t,c) \geq rt$. Then for any integer $3 \leq d \leq n/17+1$,*

$$U(d,n) = \Omega(d^{2-\log_c r} n^{1+\log_c r}).$$

Proof. From Corollaries 1.4 and 2.2,

$$U(d,n') \geq \frac{1}{4}d(d-2)n' \left(\frac{n'}{16(d-1)}\right)^{\log_c r}$$

whenever n' is a multiple of $16(d-1)$ and $n'/16(d-1)$ is an integral power of c. Let m be the integer satisfying

$$16(d-1)c^m \leq n-(d-1) < 16(d-1)c^{m+1}.$$

Let $n' = 16(d-1)c^m$, so that $(n-(d-1))/c < n' \leq n-(d-1)$. Although $U(d,n)$ is not known to be monotone in n, it is true that $U(d,n) \geq U(d,n')$ (Borodin, Ruzzo, and Tompa [5, Section 3]). Hence,

$$\begin{aligned} U(d,n) &\geq U(d,n') \\ &\geq \frac{1}{4}d(d-2)n'\left(\frac{n'}{16(d-1)}\right)^{\log_c r} \\ &> \frac{1}{4}d\left(\frac{d}{3}\right)\frac{n-(d-1)}{c}\left(\frac{n-(d-1)}{16cd}\right)^{\log_c r} \\ &\geq \frac{1}{12}d^2\frac{16n}{17c}\left(\frac{n}{17cd}\right)^{\log_c r} \\ &= \Omega\left(d^2 n\left(\frac{n}{d}\right)^{\log_c r}\right). \end{aligned}$$

□

3. The Basis. Corollary 3.3 below demonstrates that $R(t,4) \geq 6t$, yielding the exponent $\log_4 6 \approx 1.29$. In order to present the techniques needed in a simpler setting, Theorem 3.1 demonstrates that $R(t,3) \geq 4t$, which yields the slightly weaker exponent $\log_3 4 \approx 1.26$. It is easy to demonstrate that $R(t,2) \leq 2t+2$, so that no nontrivial lower bound can be derived using chains of length 2 as the basis.

THEOREM 3.1. *$R(2t-1,3) \geq 8t$, for all positive integers t.*

Proof. Let U be a $(2t-1)$-reflecting sequence for $\mathcal{L}(3)$, and $G_{00} \in \mathcal{L}(3)$ be the chain with label 00. For every traversal from vertex 0 to vertex 3 that U causes on G_{00}, consider the last time at which vertex 1 is left before vertex 3 is reached. U must have an occurrence of the (contiguous) substring 00 beginning at an even index to account for this event, because the traversal is only at vertex 1 at odd times. There are at least t traversals from vertex 0 to vertex 3; mark any t of these even indices in U with the label "00". Using the analogous process, mark t even indices in U with each of "01", "10", and "11". Since no two of these marks can share the same even index, U must have at least $4t$ even indices; that is, $|U| \geq 8t$. □

The bound in Theorem 3.1 is tight: the string $1101(10001101)^t$ is a $(2t)$-reflecting sequence for $\mathcal{L}(3)$, as can be verified by applying it to the 4 labeled chains of length 3.

THEOREM 3.2. *$R(2t-1,4) \geq 12t$, for all positive integers t.*

Proof. Let U be a $(2t-1)$-reflecting sequence for $\mathcal{L}(4)$, and $G_{000} \in \mathcal{L}(4)$ be the chain with label 000. For every traversal from vertex 0 to vertex 4 that U causes on G_{000}, consider the last time at which vertex 1 is left before vertex 4 is reached. Although U need not have an occurrence of the (contiguous) substring 000, U must have an occurrence of the substring 00 beginning at an even index to account for this event. There are at least t traversals from vertex 0 to vertex 4; mark any t of these even indices in U with the label "000". Using the analogous process, mark t even indices in U with each of "001", "100", and "101". Despite the fact that two of these marks might share a single even index, it will be shown below that these $4t$ marks account for at least $3t$ distinct occurrences of the substrings 00 and 10 beginning at

TABLE 2
Substrings of U beginning at marks

Beginning at an even index with mark:	the characters of U must match:
000	00(10)*0
001	00(00)*1
100	10(10)*0
101	10(00)*1

even indices. By an analogous argument, the other four marks "010", "011", "110", and "111" account for at least $3t$ distinct occurrences of the substrings 01 and 11 beginning at even indices. Since these two sets of indices must be disjoint, U has at least $6t$ even indices; that is, $|U| \geq 12t$.

Let the marks "000" and "101" count for two points each, and the marks "001" and "100" count for one point each, so that the $4t$ marks in U account for $6t$ points overall. Either of the substrings 00 or 10 beginning at an even index will be called simply a *pair*. The remainder of this proof demonstrates that U cannot average more than two points per pair, so there must be at least $3t$ pairs. Notice that the substring of U that begins at an even index marked "000" must be of the form 00(10)*0. Table 2 lists the substrings corresponding to the other marks. Notice the following facts:

1. Any pair with two marks must be followed immediately by another occurrence of either 00 or 10; this is evident from Table 2.
2. Two pairs consecutive in U, either one marked "000" and the other marked "101", must be followed immediately by a third occurrence of either 00 or 10, again evident from Table 2.
3. Two pairs with the same mark must be separated by a distance of at least 8, to account for two reflections.

The first case to consider is two pairs consecutive in U, each with two marks. By fact (1), these must be followed immediately by a third occurrence of either 00 or 10. By fact (3), this third pair must be unmarked. Hence, in this case, 3 pairs accumulate 6 points.

The next case to consider is a pair with two marks, followed immediately by a pair with either of the single marks "000" or "101". By fact (2), these must be followed immediately by a third occurrence of either 00 or 10. If this third pair is marked at all, it must be the single mark "001" or "100", respectively, by fact (3). Hence, in this case again, 3 pairs accumulate at most 6 points.

Any other pair with two marks must be followed immediately either by an unmarked pair, or by a pair with either of the single marks "001" or "100". In this case, 2 pairs accumulate at most 4 points.

This leaves only pairs with single marks, in which case 1 pair accumulates at most 2 points. □

The bound in Theorem 3.2 is tight: the string $10010001(101110010001)^t$ is a $(2t)$-reflecting sequence for $\mathcal{L}(4)$, as can be verified by applying it to the 8 labeled chains of length 4.

COROLLARY 3.3. *$R(t, 4) \geq 6t$, for all positive integers t.*

Proof.

CASE 1: $t = 2k - 1$ for some integer k. Then

$$R(t,4) = R(2k-1,4) \geq 12k = 6t + 6.$$

CASE 2: $t = 2k$ for some integer k. Then

$$R(t,4) = R(2k,4) \geq R(2k-1,4) \geq 12k = 6t.$$

□

Combining Corollary 3.3 with Corollaries 2.2–2.4 yields the following lower bounds.

COROLLARY 3.4. *If n is any integral power of 4, and t is any positive integer,*

$$R(t,n) \geq tn^{\log_4 6} > tn^{1.29}.$$

COROLLARY 3.5. $U(2,n) = \Omega(n^{\log_4 6}) = \Omega(n^{1.29})$.

COROLLARY 3.6. *For any integer $3 \leq d \leq n/17 + 1$,*

$$U(d,n) = \Omega(d^{2-\log_4 6} n^{1+\log_4 6}) = \Omega(d^{0.71} n^{2.29}).$$

Using only the simple ideas in the proof of Theorem 3.1, it is not difficult to prove that $R(t,5) \geq 8t$, but $\log_5 8$ is slightly less than $\log_4 6$. Determining the rate of growth of $R(t,5)$ is an open question, and could well lead to improved lower bounds on the lengths of universal traversal sequences.

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