Vector Balancing in Lebesgue Spaces

Victor Reis

BIRS Combinatorial and Geometric Discrepancy Workshop

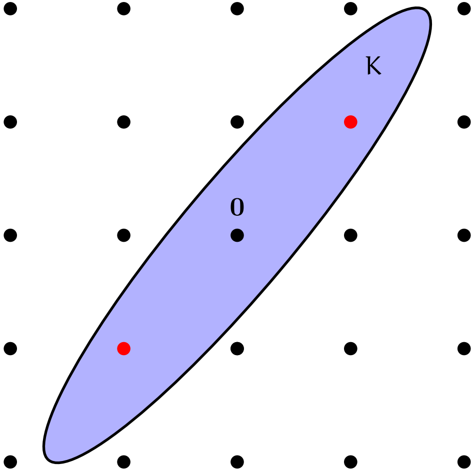
October 2020

Joint work with Thomas Rothvoss

UNIVERSITY *of* WASHINGTON

Warmup: Minkowski's Theorem (1889)

- Any symmetric convex $K \subset \mathbb{R}^n$ with volume $> 2^n$ intersects $\mathbb{Z}^n \setminus \{\mathbf{0}\}$



Non-constructive Partial Coloring

Theorem (Gluskin, 1989)

Given symmetric convex $K \subseteq \mathbb{R}^n$ with $\mathrm{vol}(K \cap [-1,1]^n) \geqslant C^n$ for $C > 1$ there exists $\boldsymbol{y} \in 2K \cap \{-1,0,1\}^n$ with $\Omega(n)$ coordinates in $\{-1,1\}$.

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- But $\mathrm{vol}(K \cap [-1,1]^n) \geqslant 2^n \gamma_n(K)$ where

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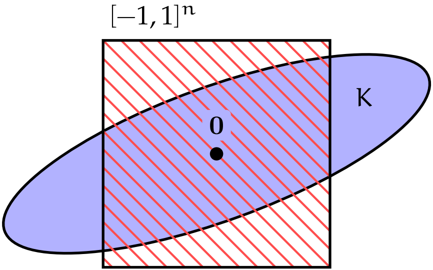
Corollary

Given symmetric convex $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geqslant 0.51^n$ there exists $\boldsymbol{y} \in 2K \cap \{-1,0,1\}^n$ with $\Omega(n)$ coordinates in $\{-1,1\}$.

Constructive Partial Coloring

Theorem (Rothvoss, FOCS 2014)

Given symmetric convex $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geqslant 0.999^n$ we can find $\boldsymbol{y} \in K \cap [-1,1]^n$ with $\Omega(n)$ coordinates in $\{-1,1\}$ in **polynomial time**.



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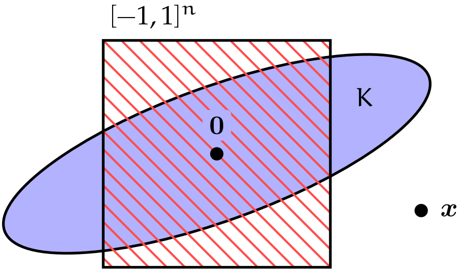

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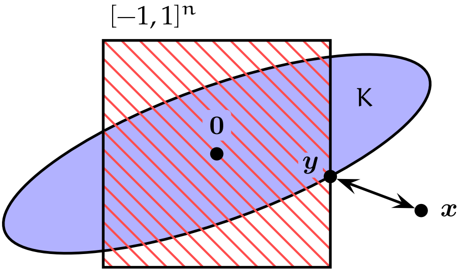
Algorithm:

(1) Sample $\boldsymbol{x} \sim N(\mathbf{0}, \boldsymbol{I}_n)$

(2) Output the projection $\boldsymbol{y}$ of $\boldsymbol{x}$ onto $K \cap [-1,1]^n$.



Constructive Partial Coloring

Theorem (R., Rothvoss '20)

For every $c \in (0,1)$, there exist $\varepsilon, \delta \geqslant \text{poly}(c)$ and a poly-time alg. that given symmetric convex $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geqslant c^n$ returns $\boldsymbol{y} \in (\frac{1}{\varepsilon}K) \cap [-1,1]^n$ with δn coordinates in $\{-1,1\}$.

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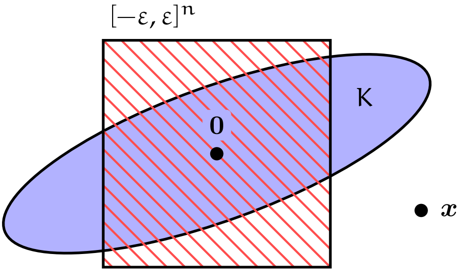

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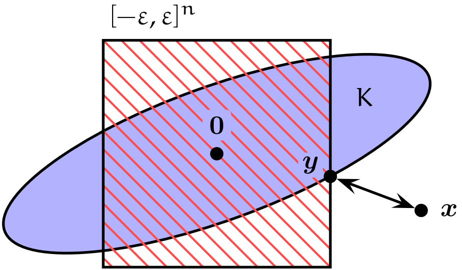

Key technical lemma

Lemma (R., Rothvoss '20)

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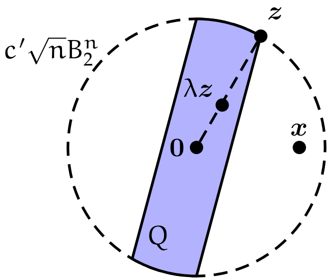

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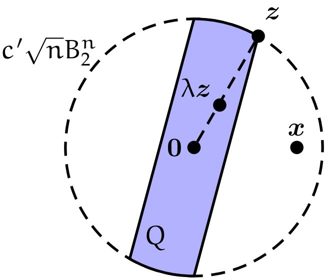

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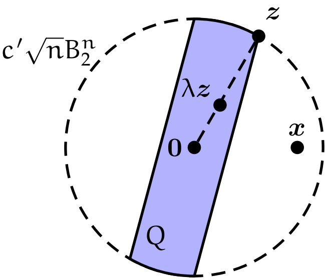

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- Upper bound $d(\boldsymbol{x}, K) \leqslant \|\boldsymbol{x} - \lambda \boldsymbol{z}\|_2$ for $\lambda = \text{poly}(c)$.

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- Proof uses polarity, Kubota integral formula and M-ellipsoids

Application: vector balancing in ℓ_p norms

- Recall $\|\boldsymbol{a}\|_p := (|a_1|^p + \cdots + |a_n|^p)^{1/p}$ (think $p \in [2, \underbrace{\log n}_{\approx\infty}]$)

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Given $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m$ with $\|\boldsymbol{a}_i\|_p \leqslant 1$, find the minimum $C_p(m, n)$ for which there always exist signs $\boldsymbol{x} \in \{-1, 1\}^n$ with

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Theorem (Spencer, 1985)

$C_\infty(m,n) \lesssim \sqrt{n \log(2m/n)}$.

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- Combine Khintchine with an $\ell_{p_0} \to \ell_\infty$ measure bound via

$$n^{1/p_0} B_{p_0}^m \cap B_\infty^m \subset n^{1/p} B_p^m$$

Measure bound for $\ell_p \to \ell_\infty$ balancing

Key Lemma (Measure bound for $\ell_p \to \ell_\infty$)

Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in B_p^m$ and rows $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathbb{R}^n$, the set $K := \{\boldsymbol{x} \in \mathbb{R}^n : \|\sum_{i=1}^n x_i \boldsymbol{a}_i\|_\infty \leqslant \sqrt{p} \cdot n^{\frac{1}{2} - \frac{1}{p}}\}$ has

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$$\gamma_n(\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{x}, \boldsymbol{v} \rangle| \leqslant 1\}) \geqslant 1 - \exp(-\|\boldsymbol{v}\|_2^{-2}/2)$$

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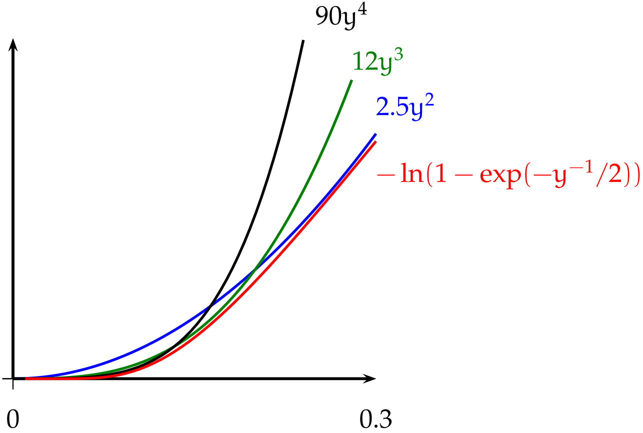
and multiply!

Key inequality

$$-\ln(1-\exp(-y^{-1}/2)) \lesssim p^{p/2} y^{p/2}$$



Last step: bootstrapping

- Combining the previous Lemma with Khintchine we can show

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- For larger p, we can take $p_0 := \max(2, 2\log(m/n))$
- In any case we get our improved measure lower bound:

$$\gamma_n\left(\left\{\boldsymbol{x} \in \mathbb{R}^n : \Big\| \sum_{i=1}^{n} x_i \boldsymbol{a}_i \Big\|_p \leqslant \sqrt{n \cdot \min\left(p, \log\left(\frac{2m}{n}\right)\right)}\right\}\right) \geqslant c^n$$

Conclusion

Main theorem (R., Rothvoss' 20)

Let $n \leqslant m$ and $1 \leqslant p \leqslant q \leqslant \infty$. Given $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m$ with $\|\boldsymbol{a}_i\|_p \leqslant 1$ we have in poly-time $\boldsymbol{x} \in [-1,1]^n$ with $n/2$ coordinates in ± 1 so that

$$\Big\| \sum_{i=1}^{n} x_i \boldsymbol{a}_i \Big\|_q \lesssim \sqrt{\min\left(p, \log\left(\frac{2m}{n}\right)\right)} \cdot n^{\max(0,1/2-1/p)+1/q}.$$

Theorem: Full coloring

Let $n \leqslant m$ and $1 \leqslant p \leqslant q \leqslant \infty$ with $\max(0, 1/2 - 1/p) + 1/q > 0$. Given $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m$ with $\|\boldsymbol{a}_i\|_p \leqslant 1$ we have in poly-time $\boldsymbol{x} \in \{-1,1\}^n$ so that

$$\Big\| \sum_{i=1}^{n} x_i \boldsymbol{a}_i \Big\|_q \lesssim \frac{\sqrt{\min\left(p, \log\left(\frac{2m}{n}\right)\right)}}{\max(0, 1/2 - 1/p) + 1/q} \cdot n^{\max(0,1/2-1/p)+1/q}.$$

Corollary: Beck-Fiala with many ones

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Given $\boldsymbol{A} \in \{0,1\}^{m \times n}$ with each column having at most $t \geqslant n$ ones, we can find $\boldsymbol{x} \in \{-1,1\}^n$ so that $\|\boldsymbol{A}\boldsymbol{x}\|_\infty \lesssim \sqrt{t}$.

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- Take $p = 4$ and $q = \infty$ in previous theorem: $\|\boldsymbol{A}\boldsymbol{x}\|_\infty \lesssim n^{1/4} t^{1/4} \leqslant \sqrt{t}$.

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- Take $p = 4$ and $q = \infty$ in previous theorem: $\|\boldsymbol{A}\boldsymbol{x}\|_\infty \lesssim n^{1/4} t^{1/4} \leqslant \sqrt{t}$.
- For $t \leqslant n$ we also get an upper bound $\sqrt{t} \cdot \log(2n/t)$.

Open problems

- Generalized Komlós Conjecture:
  Let $n \leqslant m$ and $2 \leqslant p \leqslant q \leqslant \infty$. Given $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in \mathbb{R}^m$ with $\|\boldsymbol{a}_i\|_p \leqslant 1$ can we find $\boldsymbol{x} \in \{-1, 1\}^n$ so that

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- Generalized Matrix Spencer Conjecture:
  Let $n \leqslant m^2$ and $2 \leqslant p \leqslant q \leqslant \infty$. Given $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n \in \mathbb{R}^{m \times m}$ with $\|\boldsymbol{A}_i\|_p \leqslant 1$ can we find $\boldsymbol{x} \in \{-1,1\}^n$ so that

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Thanks for your attention!