A New Framework for Matrix Discrepancy: Partial Coloring Bounds via Mirror Descent

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STOC

June 2022

Joint work with Daniel Dadush (CWI) and Haotian Jiang (UW)

UNIVERSITY *of* WASHINGTON

Talk outline

- Introduction to vector balancing

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- Measure bounds via polar covering

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- Open problems

Introduction to vector balancing

- Given $A, B \subseteq \mathbb{R}^d$, $\mathrm{vb}_n(A, B) :=$ smallest $C > 0$ so that for any $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in A$ there are signs $\boldsymbol{x} \in \{\pm 1\}^n$ with

$$\sum_{i=1}^{n} x_i \boldsymbol{v}_i \in C \cdot B.$$

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- Linear algebra shows $\mathrm{vb}_n(A, B) \leqslant 2 \cdot \mathrm{vb}_d(A, B)$: w.l.o.g. $n \leqslant d$.

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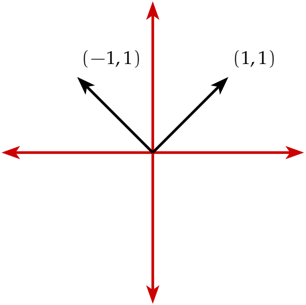

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- Conjecture (Spielman) $\mathrm{vb}_n(B_\infty^n, B_\infty^n) \leqslant 2\sqrt{n}$

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Theorem (Dadush, Jiang, R. '22)

$\mathrm{vb}_n(S_\infty^m, S_\infty^m) \lesssim \sqrt{n \log(2m^2/n)}$ for all $1 \leqslant n \leqslant m^2$.

Basic example

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \text{ and } \boldsymbol{A}_2 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

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$$\|\boldsymbol{A}_1 + \boldsymbol{A}_2\|_{S^2_\infty} = \left\| \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\|_{S^2_\infty} = 2$$

$$\|\boldsymbol{A}_1 - \boldsymbol{A}_2\|_{S^2_\infty} = \left\| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\|_{S^2_\infty} = 1$$

More interesting example

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{A}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \boldsymbol{A}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \boldsymbol{A}_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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$$\Big\| \sum_{i=1}^{4} x_i \boldsymbol{A}_i \Big\|_{S_\infty^m} \geqslant \frac{1}{\sqrt{2}} \Big\| \sum_{i=1}^{4} x_i \boldsymbol{A}_i \Big\|_{S_2^m} = \frac{1}{\sqrt{2}} \cdot \sqrt{4 \cdot 2} = 2.$$

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- Generalizes for $n = m^2$: $\sqrt{n}$ lower bound

Partial Colorings

- Given $A, B \subseteq \mathbb{R}^d$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in A$, define the *discrepancy body*

$$K := \left\{ \boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \boldsymbol{v}_i \in B \right\}$$

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- Step 3: Construct such covering via optimization

Spencer's setting

- Given $\boldsymbol{V} \in \mathbb{R}^{m \times n}$ with columns $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathsf{B}_\infty^m$, let

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Given $\boldsymbol{x} \in \mathrm{conv}(\{\boldsymbol{a}_j\}_{j \in S}) \subseteq B_2^m$ and $k \in \mathbb{N}$, there exist $i_1, \ldots, i_k \in [n]$ with

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$$N\Big(\frac{1}{\sqrt{n}}K^\circ, \frac{\sqrt{\log(2m/n)}}{\sqrt{n}}B_2^n\Big) \leqslant 2^{O(n)}$$

$$\implies \gamma_n\Big(\sqrt{n\log(2m/n)}K\Big) \geqslant 2^{-O(n)}.$$

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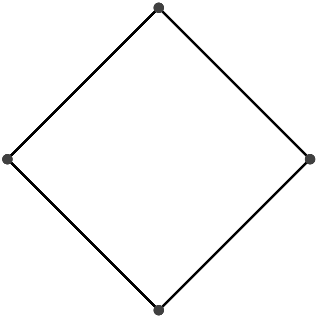

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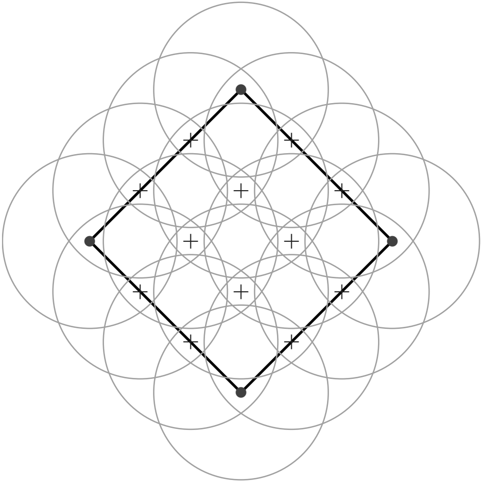

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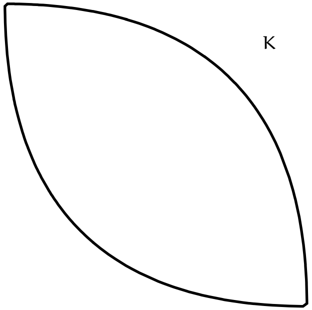

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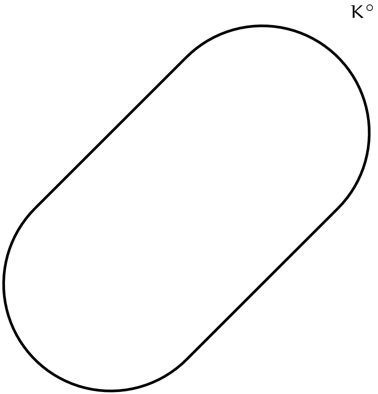

Mirror descent framework

- Fix a domain $\mathcal{D} \subseteq \mathbb{R}^m$ and a subset $\mathcal{X} \subseteq \overline{\mathcal{D}}$, start $\boldsymbol{x}_0 \in \mathcal{X} \cap \mathcal{D}$

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Mirror descent (Bubéck ’15)

After T iterations with $\eta := \frac{1}{L}\sqrt{\frac{2\rho D_\Phi(\boldsymbol{x}^*, \boldsymbol{x}_0)}{T}}$ we get a sequence $\boldsymbol{x}_t$ with

$$\min_{t<T} f(\boldsymbol{x}_t) - f(\boldsymbol{x}^*) \leqslant L \cdot \sqrt{\frac{2D_\Phi(\boldsymbol{x}^*, \boldsymbol{x}_0)}{\rho T}},$$

where $\boldsymbol{x}^* := \operatorname{argmin}_{\boldsymbol{x} \in \mathcal{X} \cap \mathcal{D}} f(\boldsymbol{x})$.

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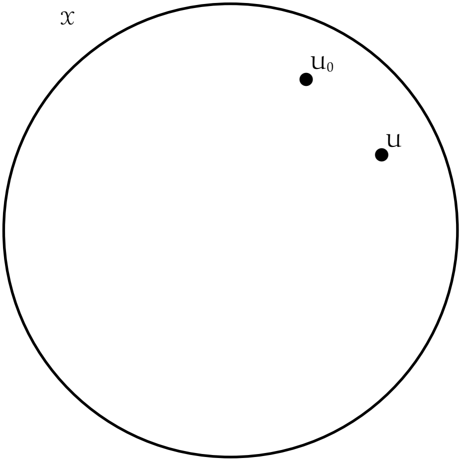

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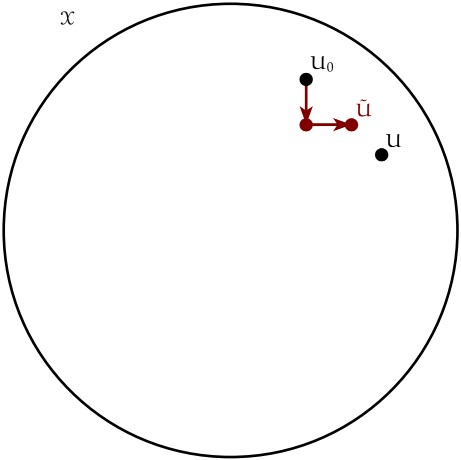

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- Recall $K^\circ = \{\underbrace{(\langle \boldsymbol{A}_1, \boldsymbol{U} \rangle, \langle \boldsymbol{A}_2, \boldsymbol{U} \rangle, \ldots, \langle \boldsymbol{A}_n, \boldsymbol{U} \rangle)}_{=: \boldsymbol{A}(\boldsymbol{U})} : \boldsymbol{U} \in S_1^m\}$ and note

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- How to bound the number of centers $\boldsymbol{A}(\tilde{\boldsymbol{U}})$ starting from some $\boldsymbol{U}_0$?

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- After $T := n$ iterations, we have at most

$$\sum_{t=0}^{n} \binom{t + 2n - 1}{2n - 1} \leqslant (n+1) \cdot \binom{3n}{n} \leqslant 2^{O(n)} \text{ centers } \boldsymbol{U}_t.$$

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giving a $\sqrt{n \log n}$ partial coloring bound.

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- Pick $2^{O(n)}$ starting points $\boldsymbol{U}_0$, total of $2^{O(n)} \cdot 2^{O(n)} \leqslant 2^{O(n)}$ centers!

Quantum relative entropy net

Key lemma

Let $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{X}$ satisfy $\|\boldsymbol{X} - \boldsymbol{Y}\|_{S_\infty^m} \leqslant \varepsilon$ for some $\varepsilon \geqslant 1/m$. Then

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Covering S_1^m with S_∞^m [HPV17]

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Further applications

Theorem (Dadush, Jiang, R. '22)

$\mathrm{vb}_n(S_\infty^{m,h}, S_\infty^{m,h}) \lesssim \sqrt{n \log(hm/n)}$ for h-block diagonal matrices.

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Thanks for your attention!