The Subspace Flatness Conjecture and Faster Integer Programming

Victor Reis

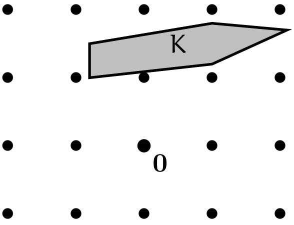
Joint work with Thomas Rothvoss

UNIVERSITY of WASHINGTON

Brief History of Integer Programming

Problem

Given a convex body $K \subseteq \mathbb{R}^n$, find a point in $K \cap \mathbb{Z}^n$ or certify $K \cap \mathbb{Z}^n = \emptyset$.



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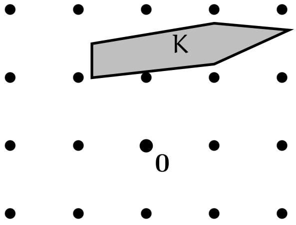

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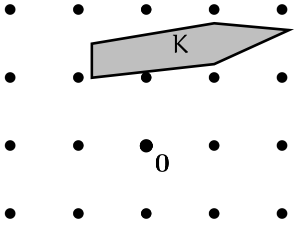

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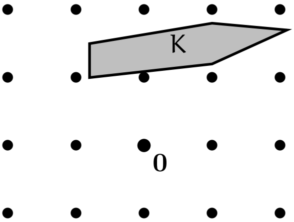

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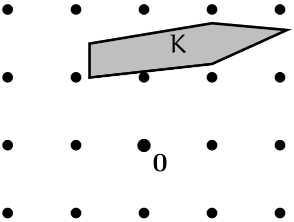

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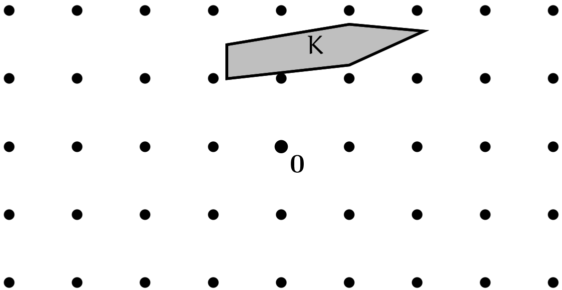
- $(O(n))^n$ [Dadush '12, Dadush, Eisenbrand, Rothvoss '22]

Dadush's algorithm



- A **lattice** is the integer span $\mathcal{L} = B\mathbb{Z}^k$ of the columns of $B \in \mathbb{R}^{n \times k}$.

Input: $K \subseteq \mathbb{R}^n$, lattice $\mathcal{L}$
Output: Point in $K \cap \mathcal{L}$

(1) Shrink K if 'too large' compared to $\mathcal{L}$
(2) Find subspace W for which $X := \Pi_W(K) \cap \Pi_W(\mathcal{L})$ is small
(3) Enumerate X
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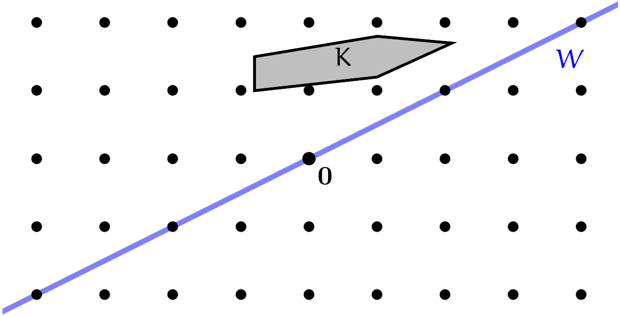

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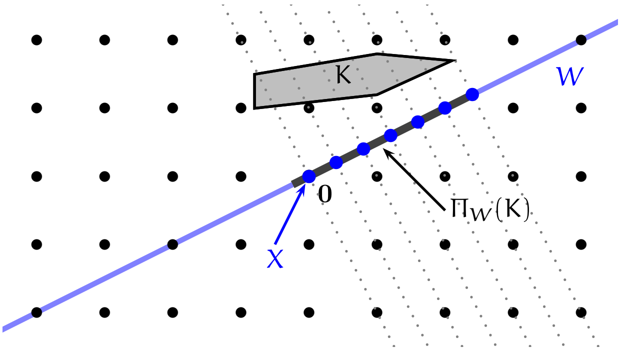

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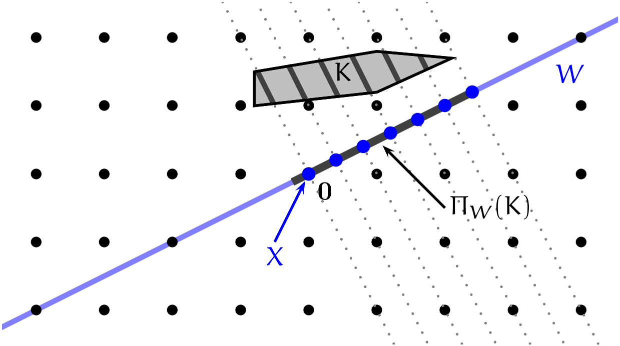

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Covering radius

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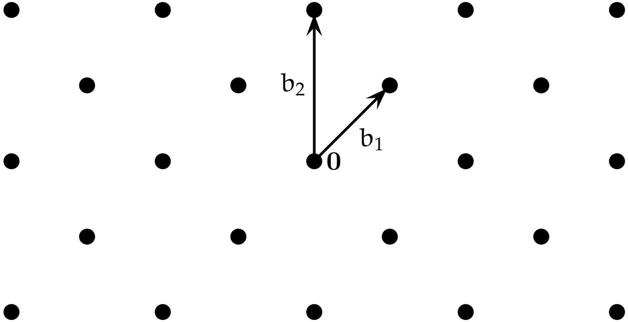
$$\mu(\mathcal{L}, K) \quad = \quad \min\{r > 0 \mid \mathcal{L} + rK = \mathbb{R}^n\}$$



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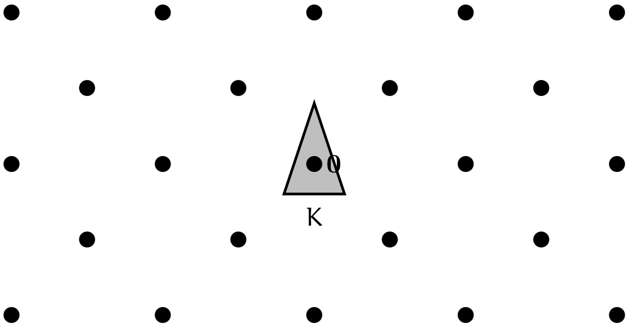

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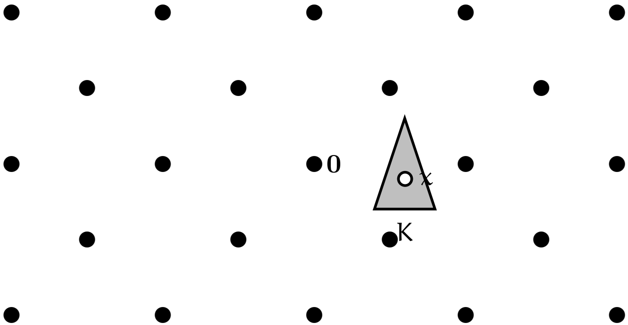

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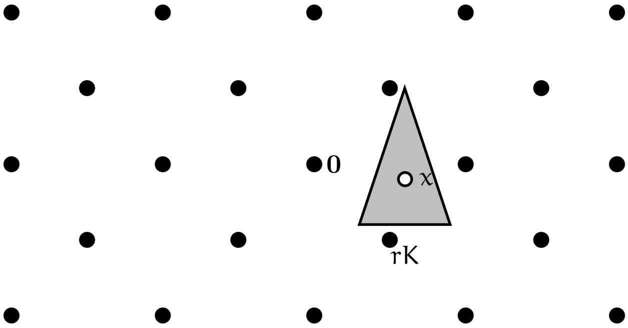

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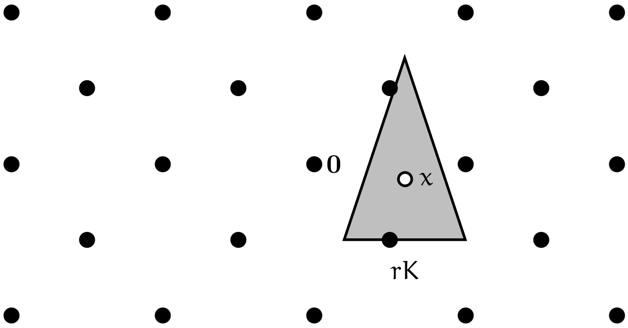

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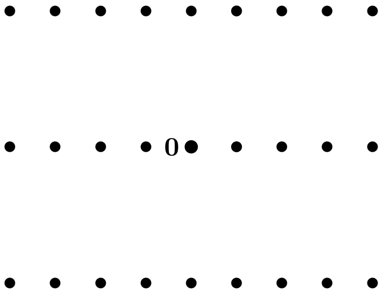

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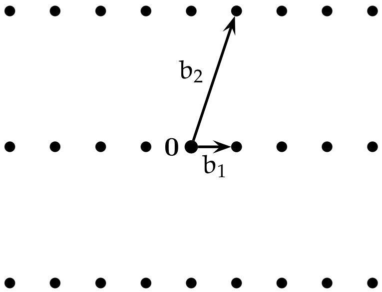

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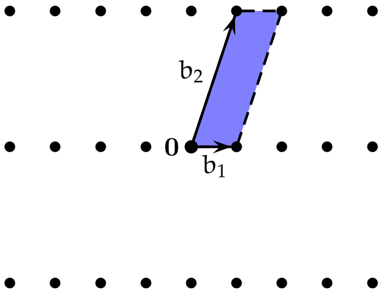

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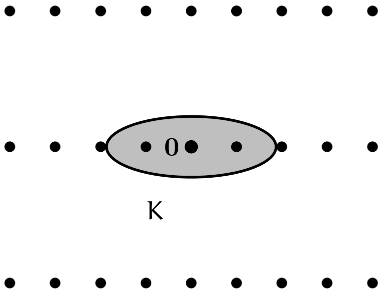

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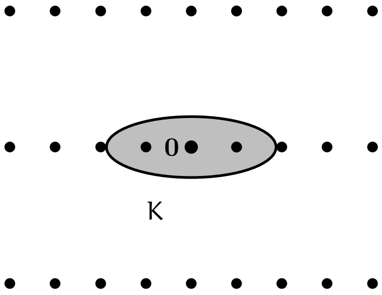

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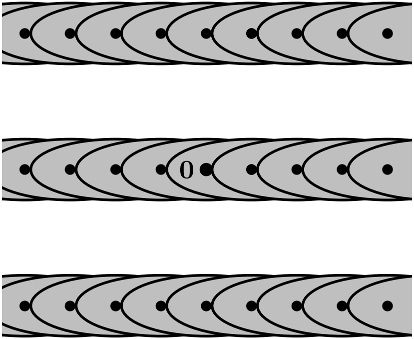

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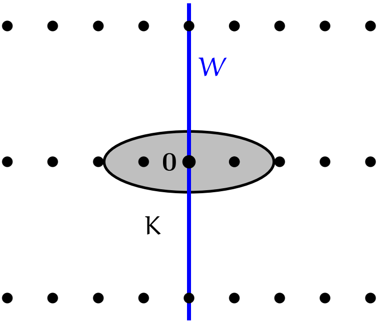

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- For any subspace $\mu(\mathcal{L}, K) \geqslant \mu(\Pi_W(\mathcal{L}), \Pi_W(K)) \geqslant (\frac{\det(\Pi_W(\mathcal{L}))}{\mathrm{vol}(\Pi_W(K))})^{1/d}$

Main results

Theorem (R., Rothvoss '23)

Given a full rank lattice $\mathcal{L} \subseteq \mathbb{R}^n$ and convex $K \subseteq \mathbb{R}^n$, recall

$$\mu(\mathcal{L}, K) = \min\{r > 0 \mid \mathcal{L} + rK = \mathbb{R}^n\}$$

There exists a subspace $W \subseteq \mathbb{R}^n$ so that

$$\left(\frac{\det(\Pi_W(\mathcal{L}))}{\mathrm{vol}(\Pi_W(K))}\right)^{1/\dim(W)} \leqslant \mu(\mathcal{L}, K) \lesssim \log^3(n) \cdot \left(\frac{\det(\Pi_W(\mathcal{L}))}{\mathrm{vol}(\Pi_W(K))}\right)^{1/\dim(W)}.$$

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Theorem (R., Rothvoss '23, following Dadush '12)

For convex $K \subseteq \mathbb{R}^n$, we can find a point in $K \cap \mathbb{Z}^n$ in time $(O(\log n))^{4n}$.

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$$\mu(\mathcal{L}, K) \leqslant \mu(\mathcal{L} \cap U, K \cap U) + \mu(\Pi_{U^\perp}(\mathcal{L}), \Pi_{U^\perp}(K))$$

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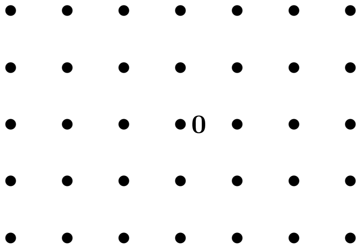
- Find 'nice' chain $\{\mathbf{0}\} = \mathcal{L}_0 \subset \cdots \subset \mathcal{L}_k = \mathcal{L}$ and take $U := \mathrm{span}(\mathcal{L}_{i^*})$

Quotient lattices

Definition

Consider a lattice $\mathcal{L} \subseteq \mathbb{R}^n$ with a sublattice $\mathcal{L}' := \mathcal{L} \cap \mathsf{U}$.
The **quotient lattice** is $\mathcal{L}/\mathcal{L}' := \Pi_{\mathsf{U}^\perp}(\mathcal{L})$.



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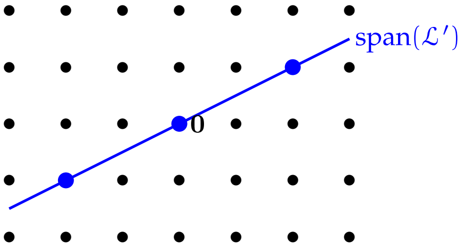

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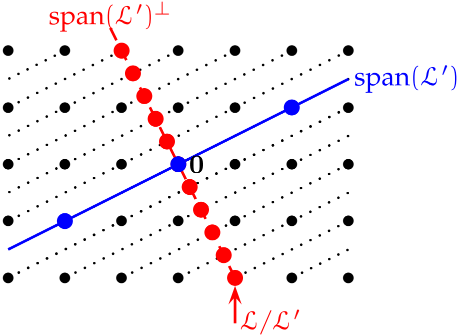

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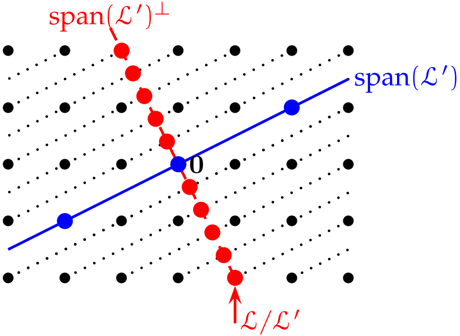

- **Intuition:** We can factor $\mathcal{L}$ into $\mathcal{L}'$ and $\mathcal{L}/\mathcal{L}'$
- For example $\det(\mathcal{L}) = \det(\mathcal{L}') \cdot \det(\mathcal{L}/\mathcal{L}')$.

The canonical filtration

- For lattice $\mathcal{L} \subseteq \mathbb{R}^n$, consider **canonical plot**

$$Q := \left\{ \big(\mathrm{rank}(\mathcal{L}'), \ln(\det(\mathcal{L}'))\big) \mid \text{sublattice } \mathcal{L}' \subseteq \mathcal{L} \right\}$$

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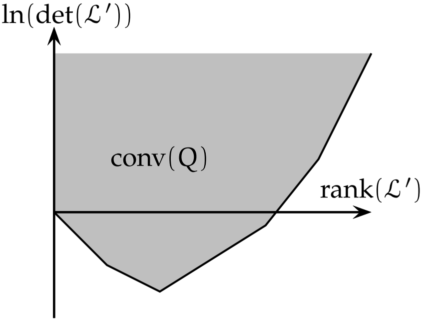

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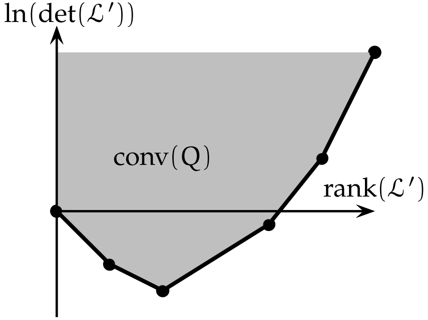
- Lower envelope of $\text{conv}(Q)$ is called **canonical polygon**



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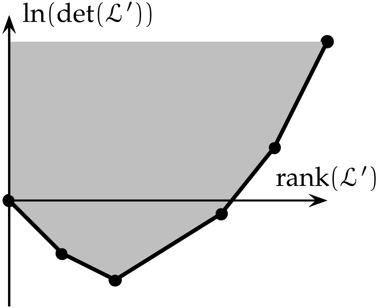

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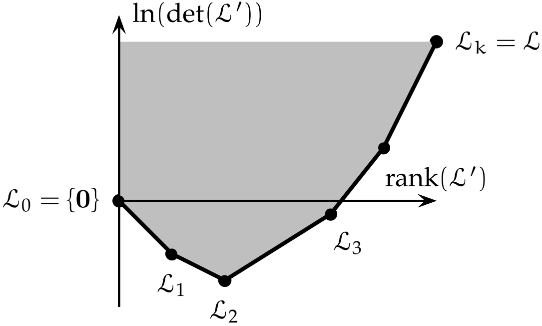

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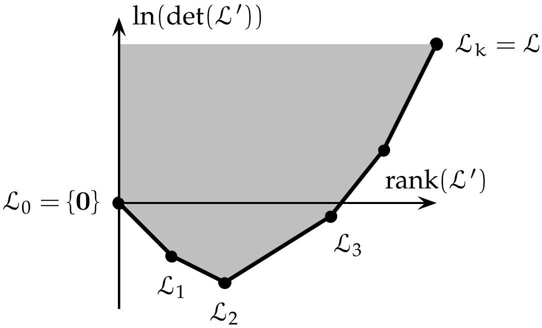

Theorem (Canonical filtration)

(a) *The vertices of the canonical plot form a chain*

$$\{\mathbf{0}\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \ldots \subset \mathcal{L}_k = \mathcal{L}.$$

(b) $r_i := \det(\mathcal{L}_i/\mathcal{L}_{i-1})^{1/\mathit{rank}(\mathcal{L}_i/\mathcal{L}_{i-1})}$ *satisfy* $r_1 < \ldots < r_k$

(c) *Each* $\frac{1}{r_i}(\mathcal{L}_i/\mathcal{L}_{i-1})$ *has determinant* $= 1$ *and its sublattices have det* $\geqslant 1$.

The Reverse Minkowski Theorem

Reverse Minkowski (Regev, Stephens-Davidowitz '17)

Let $\mathcal{L} \subseteq \mathbb{R}^n$ with $\det(\mathcal{L}') \geqslant 1$ for all $\mathcal{L}' \subseteq \mathcal{L}$. Then for $s := C \log n$,

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Definition

A lattice $\mathcal{L}$ is called <u>stable</u> if $\det(\mathcal{L}) = 1$ and $\det(\mathcal{L}') \geqslant 1$ for all $\mathcal{L}' \subseteq \mathcal{L}$.

Main proof

Take $U := \mathrm{span}(\mathcal{L}_{i^*})$ for some $i^* \in [k]$ and bound

$$\mu(\mathcal{L} \cap U, K \cap U) \leqslant \sum_{i=1}^{i^*} \mu\Big(\mathcal{L}_i/\mathcal{L}_{i-1} , \Pi_{\mathrm{span}(\mathcal{L}_{i-1})^\perp}(K \cap \mathrm{span}(\mathcal{L}_i))\Big)$$

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Lemma

For symmetric convex K and stable lattice $\mathcal{L}$ one has

$$\mu(\mathcal{L}, K) \lesssim \log(n) \cdot \ell_K,$$

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- Therefore $\mu(\mathcal{L} \cap U, K \cap U) \lesssim \log(n) \cdot \ell_K \cdot \sum_{i=1}^{i^*} r_i$.

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- $\mu(\mathcal{L} \cap U, K \cap U) \lesssim \log(n) \cdot \ell_K \cdot \sum_{i=1}^{i^*} r_i.$
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$$r_{i^*} \leqslant \det(\Pi_W(\mathcal{L}))^{1/d}$$

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- Use $r_{i^*} < \cdots < r_k$
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- Use previous lemma $\ell_{K^\circ \cap W} \leqslant \ell_{K^\circ}$

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Theorem (Figiel, Tomczak-Jaegerman, Pisier)

For any symmetric convex body $K \subseteq \mathbb{R}^n$*, there is an invertible linear map* $T : \mathbb{R}^n \to \mathbb{R}^n$ *so that* $\ell_{T(K)} \cdot \ell_{(T(K))^\circ} \lesssim n \log(n)$.

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- Suffices to pick minimal i^* so that $\dim(U) \geqslant \frac{n}{2} \implies d \geqslant \frac{n}{2}$

Putting everything together

Theorem (R., Rothvoss '23)

Given a full rank lattice $\mathcal{L} \subseteq \mathbb{R}^n$ and symmetric convex $K \subseteq \mathbb{R}^n$, recall

$$\mu(\mathcal{L}, K) = \min\{r > 0 \mid \mathcal{L} + rK = \mathbb{R}^n\}$$

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- Remains to note $\log^2(n) + \log^3(n/2) \leqslant \log^3(n)$.

Generalization to non-symmetric K

- Translate K so that the barycenter of K° is at $\mathbf{0}$
- We need:

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- This is a polar version of Rudelson's inequality:

$$\mathrm{vol}((K - K) \cap W)^{1/d} \lesssim \frac{n}{d} \cdot \max_{x \in \mathbb{R}^n} \left\{ \mathrm{vol}(K \cap (x + W))^{1/d} \right\}$$

Other implications

Theorem (R., Rothvoss '23)

For full rank lattice $\mathcal{L} \subseteq \mathbb{R}^n$ and convex body $K \subseteq \mathbb{R}^n$ one has

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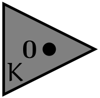

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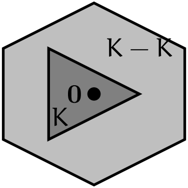

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For convex body $K \subseteq \mathbb{R}^n$ with $K \cap \mathbb{Z}^n = \emptyset$, there is a direction $c \in \mathbb{Z}^n$ with

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- Previously best known: $\lesssim n^{4/3} \log^{O(1)} n$ [Rudelson '98]

Open problem

IP in exponential time

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Thanks for your attention!