Optimal Online Discrepancy Minimization

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Joint with Janardhan Kulkarni and Thomas Rothvoss

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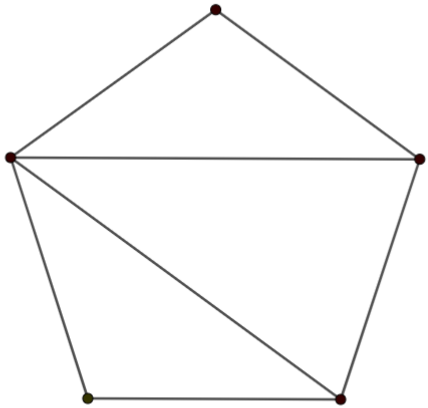
Warmup: edge orientation



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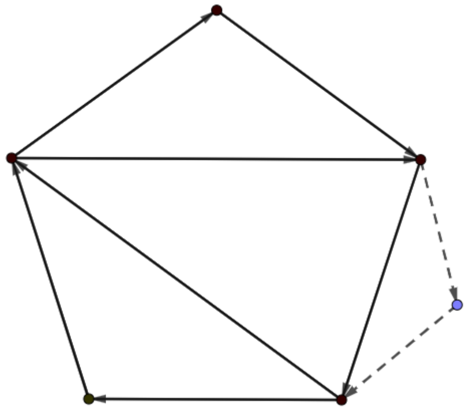

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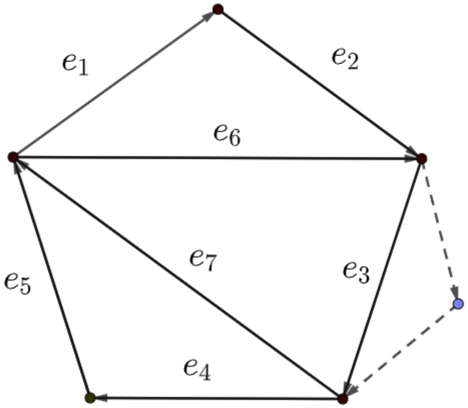

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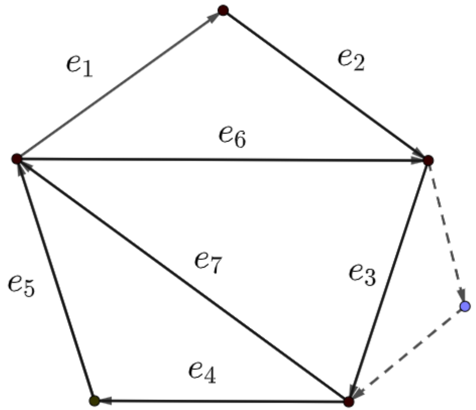

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
P_1	-1	0	0	0	1	-1	-1
P_2	1	-1	0	0	0	0	0
P_3	0	1	-1	0	0	1	0
P_4	0	0	1	-1	0	0	1
P_5	0	0	0	1	-1	0	0

Beck-Fiala Theorem (1981)

For any vectors $v_1, \dots, v_T \in [-1, 1]^n$ with at most d nonzeros each,

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for some choice of signs $x_1, \dots, x_T \in \{\pm 1\}$.

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- Best known bound $O(\sqrt{\log \min(n, T)})$ (Banaszczyk '98, BDG '16)

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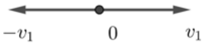

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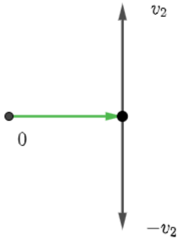

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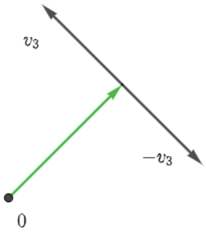

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- *Adaptive* adversary can always pick v_t so that $\|\sum_{i=1}^{t} x_i v_i\|_2 \geqslant \sqrt{T}$
- Player can also ensure $\leqslant \sqrt{T}$

Example II: Spencer's hyperbolic cosine algorithm

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- Matching lower bound $\Omega(\sqrt{n\log(2n)})$ for $T = n$

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- Player still receives one at a time and must pick signs online
- If player deterministic, same as adaptive
- What if player can use randomization?

Special case: edge orientation [Kalai '01]

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Corollary [ALS '20]

All prefix sums $\|\sum_{i=1}^{t} x_i v_i\|_\infty \leqslant O(\log(nT))$ with high probability.

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Technical: construct M_σ so that $\Pr[M_\sigma(x) = 0] \leqslant e^{-\sigma^2}$ for all $x \in \mathbb{R}$.

Our contribution

Theorem [Kulkarni, R., Rothvoss '23]

For $\|v_t\|_2 \leqslant 1$, there is an online algorithm against an oblivious adversary which keeps all prefix sums 10-subgaussian. In particular,

$$\|\sum_{i=1}^{t} x_i v_i\|_\infty \leqslant O(\sqrt{\log T}) \text{ for all } t \in [T] \text{ with high probability.}$$

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- Proof sketch: split time horizon into blocks of size $k := \Theta(\log T)$

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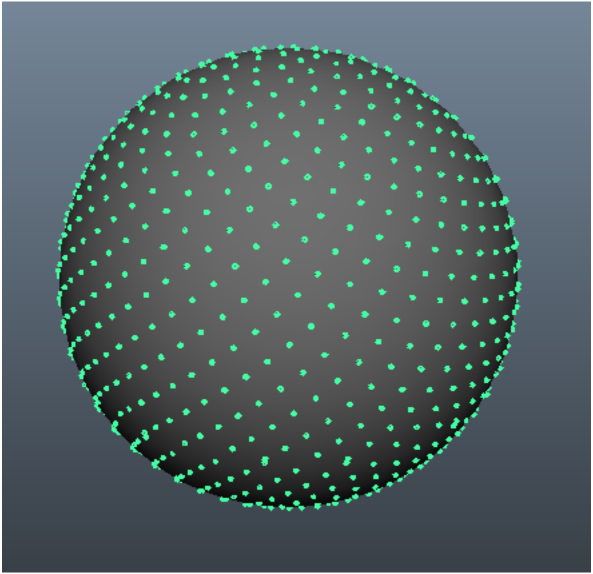
- One of the blocks will succeed with probability $1 - (1 - 2^{-k})^{T/k}$. □

ε-nets

- $P \subseteq \mathbb{R}^n$ so that, for all $\|v\|_2 \leqslant 1$, there is $p \in P$ with $\|p - v\|_2 \leqslant \varepsilon$.



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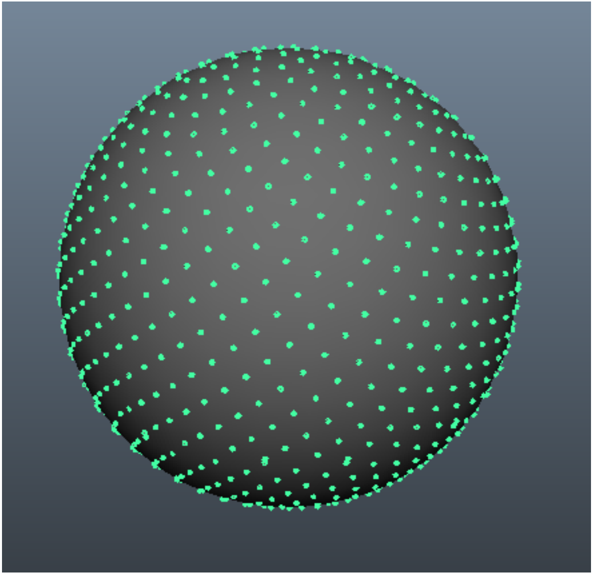

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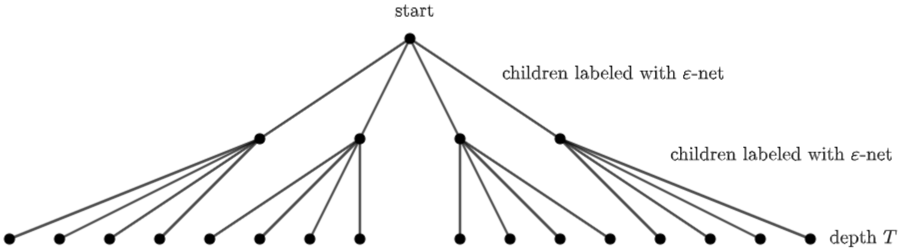

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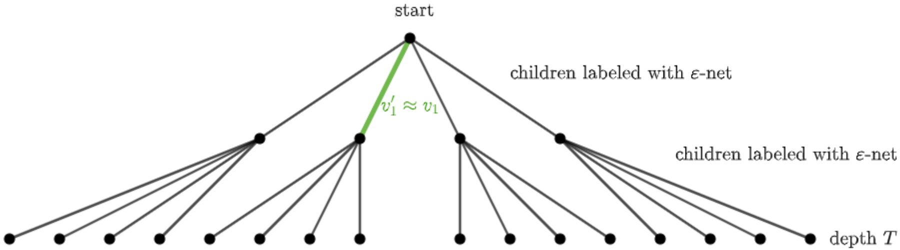

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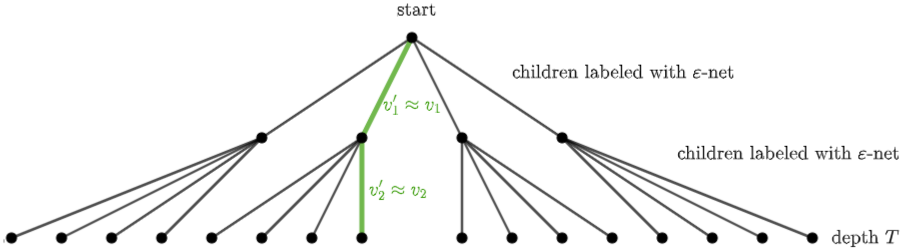

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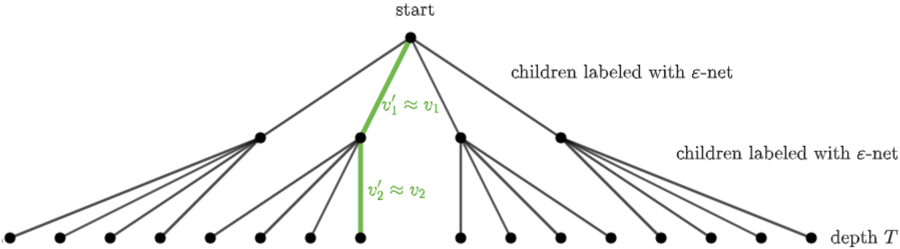

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- Show by induction $\gamma(K_t) \geqslant 1 - \frac{T-t+1}{2T}$, then iteratively find $x_1, \ldots, x_T$

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- Analogous proof with $K_i := \left(\bigcap_{j \in \text{children}_i} (K_j * v_{\{i,j\}}) \right) \cap K$.

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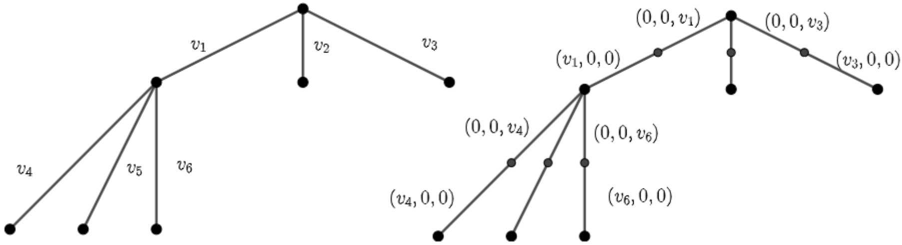

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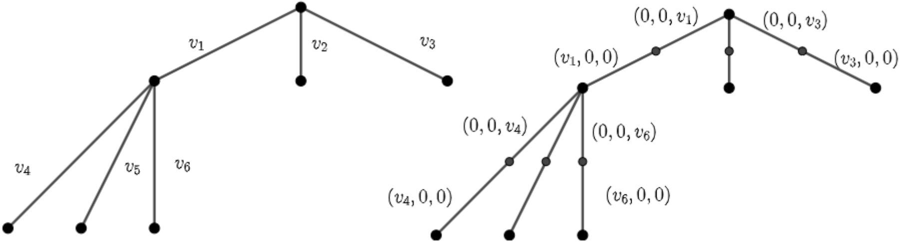

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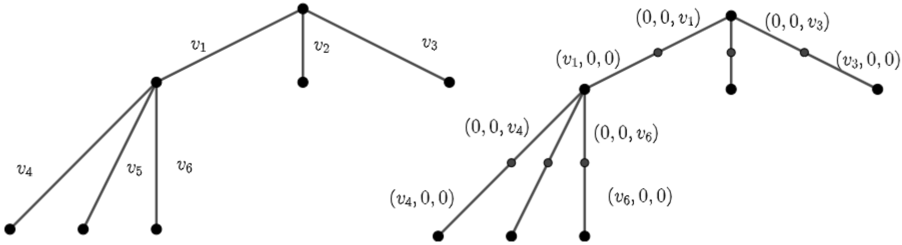

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- Define a convex body K and show $\gamma_{Nn}(K) \geqslant 1 - \frac{1}{N^{1+\delta}} \geqslant 1 - \frac{1}{2N|E|}$

Body of subgaussian distributions

- Take any $C > 2$ and define

$$K := \left\{ (y^{(1)}, \ldots, y^{(N)}) \in \mathbb{R}^{Nn} \mid Y \sim \{y^{(1)}, \ldots, y^{(N)}\} \text{ is } C\text{-subgaussian} \right\}.$$

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Lemma

Let $p \geqslant 2$ and $X_1, \ldots, X_N$ be centered, indep. r.v.'s with $\mathbb{E}[|X_i|^p] = O_p(1)$. Then

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Rosenthal '70

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- Subgaussian norm is $4.999 \cdot (2 + \delta) < 10$.

Open problems

Polynomial time algorithm

Given oblivious $v_1, \dots, v_T \in \mathbb{R}^n$ with $\|v_t\|_2 \leqslant 1$, does there exist a polynomial time online algorithm against an oblivious adversary which keeps all signed prefix sums $O(1)$-subgaussian?

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- Main theorem: $O(\sqrt{\log T})$, also $\Omega(\sqrt[3]{\log \min(n, T)})$ [AANRSW'98]

Oblivious Spencer

Given oblivious $v_1, \ldots, v_n \in [-1, 1]^n$, can we find online signs $x_1, \ldots, x_n \in \{\pm 1\}$ so that $\|\sum_{i=1}^{n} x_i v_i\|_\infty \leqslant O(\sqrt{n})$ w.h.p.?

Thanks for your attention!