An Elementary Exposition of Pisier's Inequality

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Joint work with Siddarth Iyer, Anup Rao, Thomas Rothvoss, Amir Yehudayoff

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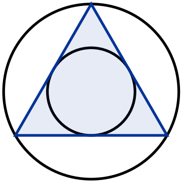
Motivation: John's Theorem (1948)

- For any convex $K \subset \mathbb{R}^m$ there exists affine linear $T : \mathbb{R}^m \to \mathbb{R}^m$ with

$$B_2^m \subseteq T(K) \subseteq m \cdot B_2^m$$



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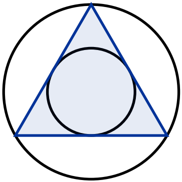

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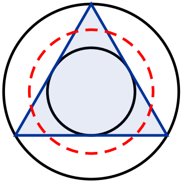

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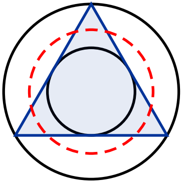

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- Write $w(K) := \mathbb{E}_{\theta \sim S^{m-1}}[\max_{v \in K} \langle v, \theta \rangle - \min_{v \in K} \langle v, \theta \rangle]$

$$w(B_2^m) \simeq \sqrt{m}$$

$$w(\Delta) \simeq \sqrt{m \log(m+1)}.$$

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- Main technical tools: Lewis ellipsoid and **Pisier's inequality.**

Fourier analysis recap

- Every $f : \{\pm 1\}^n \to \mathbb{R}$ may be written as

$$f(x) = f(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} \underbrace{\hat{f}(S)}_{\in \mathbb{R}} \prod_{j \in S} x_j$$

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Fourier analysis recap: vector-valued f

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- Bourgain 1984: $\Omega(\log m)$ lower bound (non-constructive)
- We show a $\Omega(\log m / \log \log m)$ explicit construction

Applications in CS theory

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- Deterministic volume computation [DV12]

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- $\log^{5/2}(n)$−approximation for hereditary discrepancy [DNTT18]

$$A \in \mathbb{R}^{m \times n}$$

$$\mathrm{disc}(A) := \min_{x \in \{\pm 1\}^n} \|Ax\|_\infty$$

$$\mathrm{herdisc}(A) := \max_{S \subseteq [n]} \mathrm{disc}(A_S)$$

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- Use convexity of $\| \cdot \|$ + Cauchy-Schwarz

Proof overview

- Write $x_1 + \cdots + x_n = L = P + R$ and bound

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- Bound the second term via John's theorem: ℓ_∞-estimate

$$\mathbb{E}[\|(f * R)(x)\|^2]^{1/2} \leqslant \sqrt{m} \cdot \max_{S \subseteq [n]} |\hat{R}(S)| \cdot \mathbb{E}[\|f(x)\|^2]^{1/2}.$$

Constructing the proxy: intuition

- We want to write $x_1 + \cdots + x_n = P + R$ where

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- But would need to make sure $t^1 \approx 1$, $t^0, t^2, \dots, t^k \approx 0$ for small k

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- Figiel: existence proof using heavy machinery

Constructing the proxy explicitly

Key technical lemma

Let $\ell > 0$ be odd. There is a (finitely supported) distribution $\theta \sim \mathcal{D}$ with

$$\mathop{\mathbb{E}}_{\theta \sim \mathcal{D}}[\phi(\theta) \cdot \sin^k(\theta)] = \begin{cases} 1 \text{ if } k = 1, \\ 0 \text{ if } k \in \{0, 2, 3 \ldots, \ell\} \end{cases}$$

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- In fact, for our construction ϕ:

$$\mathop{\mathbb{E}}_{\theta\sim\mathcal{D}}[|\phi(\theta)|] = \ell + \frac{1}{\ell} - \frac{1}{2\ell - 1} \leqslant \ell + \frac{1}{2\ell}.$$

Proof sketch

- Our construction is given by

$$\phi(\theta) := \frac{2\ell - 1}{\ell} \cdot \frac{\sin(\ell\theta)}{\sin^2(\theta)},$$

with $\theta \sim \mathcal{D} := \{\frac{\pi}{2\ell}, 2 \cdot \frac{\pi}{2\ell}, \ldots, (2\ell - 1) \cdot \frac{\pi}{2\ell}, (2\ell + 1) \cdot \frac{\pi}{2\ell}, \ldots, (4\ell - 1) \cdot \frac{\pi}{2\ell}\}$.

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$$\sum_{\theta \in \mathcal{D} \cup \{0,\pi\}} e^{ia\theta} = \begin{cases} 4\ell \text{ if } a \equiv 0 \mod 4\ell, \\ 0 \text{ otherwise.} \end{cases}$$

ℓ_1-estimate

- Our construction is given by

$$\phi(\theta) := \frac{2\ell - 1}{\ell} \cdot \frac{\sin(\ell\theta)}{\sin^2(\theta)},$$

with $\theta \sim \mathcal{D} := \{\frac{\pi}{2\ell}, 2 \cdot \frac{\pi}{2\ell}, \ldots, (2\ell - 1) \cdot \frac{\pi}{2\ell}, (2\ell + 1) \cdot \frac{\pi}{2\ell}, \ldots, (4\ell - 1) \cdot \frac{\pi}{2\ell}\}$.

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- Actually there is a closed formula:

$$\mathbb{E}[|\phi(\theta)|] = 4 \cdot \frac{1}{4\ell - 2} \cdot \frac{2\ell - 1}{\ell} \cdot \sum_{j=1}^{\ell} \frac{|\sin(\pi j/2)|}{\sin^2(\pi j / 2\ell)} - \frac{2}{4\ell - 2} = \ell + \frac{1}{\ell} - \frac{1}{2\ell - 1}.$$

Constructing the linear proxy

Corollary

For every odd $\ell > 0$, there exist $P, R : \{\pm 1\}^n \to \mathbb{R}$ so that $L = P + R$ and

$$\mathbb{E}[|P(x)|] \leqslant 8\ell$$

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Indeed, we may define

$$P(x) = 2 \cdot \mathop{\mathbb{E}}_{\theta \sim \mathcal{D}} \left[\phi(\theta) \cdot \prod_{j=1}^{n} \left(1 + \frac{\sin(\theta) \cdot x_j}{2} \right) \right].$$

Finishing the proof

Putting the pieces together: take $\ell = \frac{1}{2}\log_2(m+1) + C$ to be odd $\in \mathbb{N}$

$$\begin{aligned}
\mathbb{E}[\|f_{\text{lin}}(x)\|^2]^{1/2} &\leqslant \Big(\mathbb{E}[\|P(x)\|] + \sqrt{m}\cdot \max_{S\subseteq[n]} |\hat{R}(S)|\Big)\cdot \mathbb{E}[\|f(x)\|^2]^{1/2} \\
&\leqslant 8\ell\left(1+\frac{\sqrt{m}}{2^\ell}\right)\cdot \mathbb{E}[\|f(x)\|^2]^{1/2} \\
&\lesssim \log(m+1)\cdot \mathbb{E}[\|f(x)\|^2]^{1/2}.
\end{aligned}$$

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Suppose for all $n \in \mathbb{N}$ we can find $F : \{\pm 1\}^n \to \mathbb{R}$ such that

$$\|F\|_\infty \leqslant O(1)$$

$$\hat{F}(\{j\}) = 1/\sqrt{n} \text{ for all } j \in [n]$$

$$m := |\mathrm{supp}(\hat{F})| = |\{S : \hat{F}(S) \neq 0\}| \leqslant 2^{C\sqrt{n}\log n}.$$

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- Define a norm $\|\cdot\| : \mathbb{R}^{\mathrm{supp}(\hat{F})} \to \mathbb{R}_{\geqslant 0}$ as

$$\|v\| = \max_{x \in \{\pm 1\}^n} \Big| \sum_{S \in \mathrm{supp}(\hat{F})} v_S \prod_{j \in S} x_j \Big|$$

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- Take

$$H(x) := \operatorname{Im} \prod_{j=1}^{n} \left(1 + \frac{i}{\sqrt{n}} x_j\right) = \sum_{S \subseteq [n]} \operatorname{Im}\left(\left(\frac{i}{\sqrt{n}}\right)^{|S|}\right) \cdot \prod_{j \in S} x_j$$

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Open problems

- Improved reverse Urysohn inequality.
  Show for any convex $K \subset \mathbb{R}^m$ there exists linear $T : \mathbb{R}^m \to \mathbb{R}^m$ with

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Thanks for your attention!