A Tighter Relation Between Hereditary Discrepancy & Determinant Lower Bound

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SOSA 2022

Joint work with Haotian Jiang

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Outline of the talk

- Introduction to discrepancy and the determinant lower bound

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- Summary of previous results

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- Summary of previous results
- Our contribution
- Open problems

Discrepancy as rounding

- Given $A \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^n$, how can we *round* y to $x \in \mathbb{Z}^n$ so that

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- Denote $\|z\|_\infty := \max_{i \in [n]} |z_i|$ and define the *linear discrepancy*
$$\begin{aligned} \text{lindisc}(A) &:= \max_{y \in \mathbb{R}^n} \text{lindisc}(A, y) \\ &:= \max_{y \in \mathbb{R}^n} \min_{x \in \mathbb{Z}^n} \|Ax - Ay\|_\infty \end{aligned}$$

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- (Li-Nikolov, 2020) NP-hard to compute. Can we approximate it?

Determinant lower bound

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Theorem (Lovász-Spencer-Vesztergombi, 1986)

For any square $A \in \mathbb{R}^{n \times n}$ we have $\text{lindisc}(A) \geqslant \frac{1}{2}|\det(A)|^{1/n}$.

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What about an upper bound?

Hereditary discrepancy

Define the *hereditary discrepancy*

$$\text{herdisc}(A) := \max_{S \subseteq [n]} \min_{x \in \{-1,1\}^S} \|A_S x\|_\infty,$$

where A_S is the submatrix of A with columns from S.

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$$\text{Alternatively: } \tfrac{1}{2}\mathbb{Z}^n \subseteq \tfrac{1}{2}\text{herdisc}(A) \cdot K + \mathbb{Z}^n$$

- Second step: Show this implies $\mathbb{R}^n \subseteq \text{herdisc}(A) \cdot K + \mathbb{Z}^n$.

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Second step: for any closed convex $K \subset \mathbb{R}^n$,

$$\frac{1}{2}\mathbb{Z}^n \subseteq K + \mathbb{Z}^n \implies \mathbb{R}^n \subseteq 2K + \mathbb{Z}^n.$$

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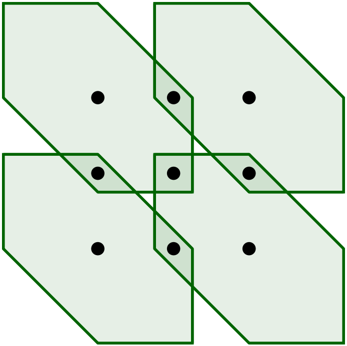

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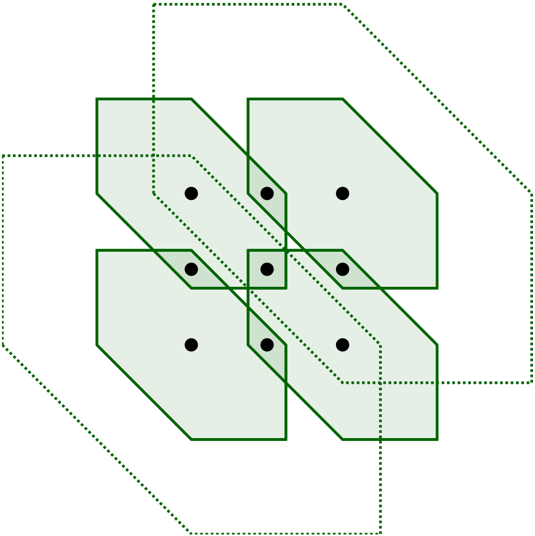

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For any $A \in \mathbb{R}^{m \times n}$ we have $\text{herdisc}(A) \geqslant \frac{1}{2} \cdot \max_{\substack{(S,T) \subseteq [m] \times [n] \\ |S|=|T|=k}} |\det(A_{S,T})|^{1/k}$.

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- Denote $\mathrm{detLB}(A) := \max_{k \in \mathbb{N}} \max_{\substack{(S,T) \subseteq [m] \times [n] \\ |S|=|T|=k}} |\det(A_{S,T})|^{1/k}$

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Theorem (Matoušek, 2011)

For any $A \in \mathbb{R}^{m \times n}$ we have $\text{herdisc}(A) \lesssim \sqrt{\log m} \cdot \log n \cdot \text{detLB}(A)$.

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Matoušek's lemma (2011)

For any $A \in \mathbb{R}^{m \times n}$ we have $\mathrm{detLB}(A) \gtrsim \mathrm{hervecdisc}(A)/\sqrt{\log n}$.

Our contribution

Theorem (Jiang-R., 2021)

For any $A \in \mathbb{R}^{m \times n}$ we have $\mathrm{herdisc}(A) \lesssim \sqrt{\log m \cdot \log n} \cdot \mathrm{detLB}(A)$.

Combination of two results involving partial *hereditary vector discrepancy*:

Theorem (Bansal, 2010), slight adaptation

For any $A \in \mathbb{R}^{m \times n}$ we have $\mathrm{herdisc}(A) \lesssim \sqrt{\log m \cdot \log n} \cdot \mathrm{herpvdisc}(A)$.

Key lemma

For any $A \in \mathbb{R}^{m \times n}$ we have $\mathrm{detLB}(A) \gtrsim \mathrm{herpvdisc}(A)$.

Partial vector discrepancy

Given $A \in \mathbb{R}^{m \times n}$, the partial vector discrepancy is given by the SDP

$$\begin{aligned}
&\min \ \lambda \\
&\Big\| \sum_{j=1}^{n} a_{ij} \boldsymbol{v}_j \Big\|_2 \leqslant \lambda \quad \forall\, i \in [m] \\
&\sum_{j=1}^{n} \|\boldsymbol{v}_j\|_2^2 \geqslant n/2 \\
&\|\boldsymbol{v}_j\|_2^2 \leqslant 1 \quad \forall\, j \in [n].
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In order to show detLB $\gtrsim \lambda$, suffices to beat any dual feasible solution

Dual partial vector discrepancy SDP

The dual SDP is given by

$$\begin{aligned} &\max \quad n\gamma - \sum_{j=1}^{n} z_j \\ &\sum_{i=1}^{m} w_i \boldsymbol{a}_i \boldsymbol{a}_i^\top + \sum_{j=1}^{n} z_j \boldsymbol{e}_j \boldsymbol{e}_j^\top \succeq 2\gamma \cdot I_n \\ &\sum_{i=1}^{m} w_i = 1 \\ &\boldsymbol{w}, \boldsymbol{z} \geqslant 0. \end{aligned}$$

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Idea: find a submatrix with large singular values, therefore large det

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$J := \{j \in [n] : z_j < 1.5\gamma\}$ so that $|J| \geqslant n/3$ and $2\gamma - z_j > 0.5\gamma$ for $j \in J$.

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It follows all eigenvalues of $\sum_{i=1}^{m} w_i \boldsymbol{a}_{i,J} \boldsymbol{a}_{i,J}^\top$ are $> 0.5\gamma \geqslant 0.5 \cdot \lambda^2/n$

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- Cauchy-Binet also gives

$$\det(\sum_{i=1}^{m} w_i \boldsymbol{a}_{i,J} \boldsymbol{a}_{i,J}^\top) = \sum_{\substack{I \subseteq [m] \\ |I|=|J|}} \det(A_{I,J})^2 \prod_{i \in I} w_i$$

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- Combining the two inequalities, $\mathrm{detLB}(A) \gtrsim \lambda \cdot \sqrt{|J|/n} \gtrsim \lambda$.

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Thanks for your attention!